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EXACT BRANE SOLUTIONS IN CURVED BACKGROUNDS

PLAMEN BOZHILOV

Department of Theoretical Physics, Konstantin Preslavsky University
9712 Shoumen, Bulgaria

Abstract. We consider the classical null p-brane dynamics in D-dimensional curved backgrounds and apply the Batalin–Fradkin–Vilkovisky approach for BRST quantization of general gauge theories. Then we develop a method for solving the tensionless p-brane equations of motion and constraints. This is possible whenever there exists at least one Killing vector for the background metric. It is shown that the same method can be also applied for the *tensile* 1-branes. Finally, we give two examples of explicit exact solutions in four dimensions.

1. Introduction

The p-brane is a p-dimensional relativistic object, which evolving in space–time describes a $(p+1)$-dimensional worldvolume. In this terminology, $p = 0$ corresponds to a point particle, $p = 1$ corresponds to a string, $p = 2$ corresponds to a membrane and so on. Every p-brane characterizes by its tension T_p with dimension of $(\mathrm{mass})^{p+1}$. When the tension $T_p = 0$, the p-brane is called null or tensionless one. This relationship between the null branes and the tensile ones generalizes the correspondence between massless and massive particles for the case of extended objects. Thus, the tensionless branes may be viewed as a high-energy limit of the tensile ones.

As is known, there exist five consistent string theories in ten dimensions: Type IIA with $N = 2$ non-chiral supersymmetry, type IIB with $N = 2$ chiral supersymmetry, type I with $N = 1$ supersymmetry and gauge symmetry $SO(32)$ and heterotic strings with $N = 1$ supersymmetry with $SO(32)$ or $E_8 \times E_8$ gauge symmetry.

The superstring dynamics unify all fundamental interactions between the elementary particles, including gravity, at super high energies. The p-branes arise naturally in the superstring theory, because there exist exact brane solutions of the superstring effective equations of motion. The 2-branes and the 5-branes are the fundamental dynamical objects in eleven dimensional M-theory, which is the strong coupling limit of the five superstring theories in ten dimensions, and which low energy field theory limit is the eleven dimensional supergravity. Particular type of 3-branes arise in the Randall-Sundrum brane world scenario.

The purpose of this paper is to present some investigations on the p-brane dynamics in curved backgrounds, which are part of the string theory backgrounds, with the aim of finding exact solutions of the equations of motion and constraints, and further application of the received results. For example, if the branes are viewed as space–time probes, the obtained exact solutions may have relevance to the singularity structure of branes. On the other hand, these solutions may have cosmological implications especially in the early universe. It is worth checking if these solutions lead to self-consistent brane cosmology. The possible application in the framework of the modern concept of **brane world universe** is especially interesting. Another appropriate field of realization of these results is the investigation of the solution properties near black hole horizons, where the tensionless limit is a good approximation and significantly simplifies the corresponding analysis. The approach of Batalin, Fradkin and Vilkovisky for BRST quantization of general gauge theories, applied to the null p-branes, gives the possibility for quantization of such systems in curved backgrounds.

2. Null Branes

2.1. Lagrangian Formulation

The action for the bosonic null p-brane in a D-dimensional curved space–time with metric tensor g_{MN} can be written in the form [1]:

$$S = \int \mathrm{d}^{p+1}\xi \mathcal{L} , \qquad \mathcal{L} = V^m V^n \partial_m X^M \partial_n X^N g_{MN} , \tag{1}$$
$$\partial_m = \partial/\partial\xi^m , \qquad \xi^m = (\xi^0, \xi^i) = (\tau, \sigma^i) ,$$
$$m, n = 0, 1, \ldots, p , \qquad i = 1, \ldots, p , \qquad M, N = 0, 1, \ldots, D-1 .$$

To prove the invariance of the action under infinitesimal diffeomorphisms on the world volume (reparametrizations), we first write down the corresponding transformation law for the (r, s)-type tensor density of weight a

$$\begin{aligned}\delta_\varepsilon T^{J_1\ldots J_r}_{K_1\ldots K_s}[a] = L_\varepsilon T^{J_1\ldots J_r}_{K_1\ldots K_s}[a] &= \varepsilon^L \partial_L T^{J_1\ldots J_r}_{K_1\ldots K_s}[a] \\ &+ T^{J_1\ldots J_r}_{KK_2\ldots K_s}[a]\partial_{K_1}\varepsilon^K + \cdots + T^{J_1\ldots J_r}_{K_1\ldots K_{s-1}K}[a]\partial_{K_s}\varepsilon^K \\ &- T^{JJ_2\ldots J_r}_{K_1\ldots K_s}[a]\partial_J\varepsilon^{J_1} - \cdots - T^{J_1\ldots J_{r-1}J}_{K_1\ldots K_s}[a]\partial_J\varepsilon^{J_r} \\ &+ aT^{J_1\ldots J_r}_{K_1\ldots K_s}[a]\partial_L\varepsilon^L\end{aligned} \tag{2}$$

where L_ε is the Lie derivative along the vector field ε. Using (2), one verifies that if $X^M(\xi)$, $g_{MN}(\xi)$ are world-volume scalars ($a = 0$) and $V^m(\xi)$ is a world-volume (1, 0)-type tensor density of weight $a = 1/2$, then $\partial_m X^N$ is a (0, 1)-type tensor, $\partial_m X^M \partial_n X^N g_{MN}$ is a (0, 2)-type tensor and $\mathcal{L}$ is a scalar density of weight $a = 1$. Therefore,

$$\delta_\varepsilon S = \int \mathrm{d}^{p+1}\xi \partial_m \left(\varepsilon^m \mathcal{L}\right)$$

and this variation vanishes under suitable boundary conditions.

The equations of motion following from (1) are:

$$\partial_m \left(V^m V^n \partial_n X^L\right) + \Gamma^L_{MN} V^m V^n \partial_m X^M \partial_n X^N = 0\,,$$
$$V^m \partial_m X^M \partial_n X^N g_{MN} = 0\,,$$

where Γ^L_{MN} is the connection compatible with the metric g_{MN}:

$$\Gamma^L_{MN} = \frac{1}{2} g^{LR} \left(\partial_M g_{NR} + \partial_N g_{MR} - \partial_R g_{MN}\right)\,.$$

For the transition to Hamiltonian picture it is convenient to rewrite the Lagrangian density (1) in the form ($\partial_\tau = \partial/\partial\tau, \partial_j = \partial/\partial\sigma^j$):

$$L = \frac{1}{4\lambda^0} g_{MN} \left(\partial_\tau - \lambda^j \partial_j\right) X^M \left(\partial_\tau - \lambda^k \partial_k\right) X^N, \tag{3}$$

where

$$V^m = (V^0, V^j) = \left(-\frac{1}{2\sqrt{\lambda^0}}, \frac{\lambda^j}{2\sqrt{\lambda^0}}\right)\,.$$

Now the equation of motion for X^N takes the form:

$$\begin{aligned}&\partial_\tau \left[\frac{1}{2\lambda^0}\left(\partial_\tau - \lambda^k \partial_k\right) X^L\right] - \partial_j \left[\frac{\lambda^j}{2\lambda^0}\left(\partial_\tau - \lambda^k \partial_k\right) X^L\right] \\ &\quad + \frac{1}{2\lambda^0}\Gamma^L_{MN}\left(\partial_\tau - \lambda^j \partial_j\right) X^M \left(\partial_\tau - \lambda^k \partial_k\right) X^N = 0\,.\end{aligned}$$

The equations of motion for the Lagrange multipliers λ^0 and λ^j which follow from (3) give the constraints:

$$g_{MN}\left(\partial_\tau - \lambda^j\partial_j\right)X^M\left(\partial_\tau - \lambda^k\partial_k\right)X^N = 0\,,$$
$$g_{MN}\left(\partial_\tau - \lambda^k\partial_k\right)X^M\partial_j X^N = 0\,.$$

In terms of X^N and the conjugated momentum P_N they read:

$$T_0 = g^{MN}P_M P_N = 0\,, \qquad T_j = P_N\partial_j X^N = 0\,. \tag{4}$$

2.2. Hamiltonian Formulation

The Hamiltonian which corresponds to the Lagrangian density (3) is a linear combination of the constraints (4):

$$H_0 = \int \mathrm{d}^p\sigma \left(\lambda^0 T_0 + \lambda^j T_j\right)\,.$$

They satisfy the following (equal τ) Poisson bracket algebra

$$\begin{aligned}
&\{T_0(\underline{\sigma}_1), T_0(\underline{\sigma}_2)\} = 0\,, \qquad \underline{\sigma} = (\sigma^1, \dots, \sigma^p)\,,\\
&\{T_0(\underline{\sigma}_1), T_j(\underline{\sigma}_2)\} = [T_0(\underline{\sigma}_1) + T_0(\underline{\sigma}_2)]\partial_j\delta^p(\underline{\sigma}_1 - \underline{\sigma}_2)\,,\\
&\{T_j(\underline{\sigma}_1), T_k(\underline{\sigma}_2)\} = [\delta^l_j T_k(\underline{\sigma}_1) + \delta^l_k T_j(\underline{\sigma}_2)]\partial_l\delta^p(\underline{\sigma}_1 - \underline{\sigma}_2)\,.
\end{aligned} \tag{5}$$

The equalities (5) show that the constraint algebra is the same for flat and for curved backgrounds. Having in mind the above algebra, one can use the Batalin–Fradkin–Vilkovisky approach for BRST quantization of general gauge theories, and to construct the corresponding BRST charge Ω (* = complex conjugation)

$$\Omega = \Omega^{\min} + \pi_m\bar{\mathcal{P}}^m\,, \quad \{\Omega, \Omega\} = 0\,, \quad \Omega^* = \Omega\,. \tag{6}$$

$\Omega^{\min}$ in (6) can be written as [1]

$$\Omega^{\min} = \int \mathrm{d}^p\sigma\{T_0\eta^0 + T_j\eta^j + \mathcal{P}_0[(\partial_j\eta^j)\eta^0 + (\partial_j\eta^0)\eta^j] + \mathcal{P}_k(\partial_j\eta^k)\eta^j\}\,,$$

and can be represented also in the form

$$\Omega^{\min} = \int \mathrm{d}^p\sigma\left[\left(T_0 + \frac{1}{2}T_0^{gh}\right)\eta^0 + \left(T_j + \frac{1}{2}T_j^{gh}\right)\eta^j\right] + \int \mathrm{d}^p\sigma\partial_j\left(\frac{1}{2}\mathcal{P}_k\eta^k\eta^j\right).$$

Here a superscript gh is used for the ghost part of the total gauge generators

$$T_m^{\mathrm{tot}} = \{\Omega, \mathcal{P}_m\} = \{\Omega^{\min}, \mathcal{P}_m\} = T_m + T_m^{gh}\,.$$

We recall that the Poisson bracket algebras of T_m^{tot} and T_m coincide for first rank systems which is the case under consideration. The manifest expressions for T_m^{gh} are:

$$\begin{aligned} T_0^{gh} &= 2\mathcal{P}_0\partial_j\eta^j + (\partial_j\mathcal{P}_0)\,\eta^j\,, \\ T_j^{gh} &= 2\mathcal{P}_0\partial_j\eta^0 + (\partial_j\mathcal{P}_0)\,\eta^0 + \mathcal{P}_j\partial_k\eta^k + \mathcal{P}_k\partial_j\eta^k + (\partial_k\mathcal{P}_j)\,\eta^k\,. \end{aligned}$$

Up to now, we introduced canonically conjugated ghosts $(\eta^m, \mathcal{P}_m)$, $(\bar{\eta}_m, \bar{\mathcal{P}}^m)$ and momenta π_m for the Lagrange multipliers λ^m in the Hamiltonian. They have Poisson brackets and Grassmann parity as follows (ϵ_m is the Grassmann parity of the corresponding constraint):

$$\begin{aligned} &\{\eta^m, \mathcal{P}_n\} = \delta_n^m, \quad \epsilon(\eta^m) = \epsilon(\mathcal{P}_m) = \epsilon_m + 1\,, \\ &\{\bar{\eta}_m, \bar{\mathcal{P}}^n\} = -(-1)^{\epsilon_m\epsilon_n}\delta_m^n\,, \qquad \epsilon(\bar{\eta}_m) = \epsilon(\bar{\mathcal{P}}^m) = \epsilon_m + 1\,, \\ &\{\lambda^m, \pi_n\} = \delta_n^m, \quad \epsilon(\lambda^m) = \epsilon(\pi_m) = \epsilon_m\,. \end{aligned}$$

The BRST-invariant Hamiltonian is

$$H_{\tilde{\chi}} = H^{\min} + \{\tilde{\chi}, \Omega\} = \{\tilde{\chi}, \Omega\}\,, \tag{7}$$

because from $H_{\text{canonical}} = 0$ it follows $H^{\min} = 0$. In this formula $\tilde{\chi}$ stands for the gauge fixing fermion ($\tilde{\chi}^* = -\tilde{\chi}$). We use the following representation for the latter

$$\tilde{\chi} = \chi^{\min} + \bar{\eta}_m\left(\chi^m + \frac{1}{2}\rho_{(m)}\pi^m\right)\,, \qquad \chi^{\min} = \lambda^m\mathcal{P}_m$$

where $\rho_{(m)}$ are scalar parameters and we have separated the π^m-dependence from χ^m. If we adopt that χ^m does not depend on the ghosts $(\eta^m, \mathcal{P}_m)$ and $(\bar{\eta}_m, \bar{\mathcal{P}}^m)$, the Hamiltonian $H_{\tilde{\chi}}$ from (7) takes the form

$$\begin{aligned} H_{\tilde{\chi}} = H_{\chi}^{\min} + \mathcal{P}_m\bar{\mathcal{P}}^m - \pi_m\left(\chi^m + \frac{1}{2}\rho_{(m)}\pi^m\right) \\ + \bar{\eta}_m\{\chi^m, T_n\}\,\eta^n\,, \end{aligned} \tag{8}$$

where

$$H_{\chi}^{\min} = \{\chi^{\min}, \Omega^{\min}\}\,.$$

One can use the representation (8) for $H_{\tilde{\chi}}$ to obtain the corresponding BRST invariant Lagrangian density

$$L_{\tilde{\chi}} = L + L_{GH} + L_{GF}\,.$$

Here L is given in (3), L_{GH} stands for the ghost part and L_{GF} — for the gauge fixing part of the Lagrangian density. The manifest expressions for L_{GH} and L_{GF} are [1]:

$$\begin{aligned}
L_{GH} = & -\partial_\tau\bar{\eta}_0\partial_\tau\eta^0 - \partial_\tau\bar{\eta}_j\partial_\tau\eta^j + \lambda^0[2\partial_\tau\bar{\eta}_0\partial_j\eta^j + (\partial_j\partial_\tau\bar{\eta}_0)\eta^j] \\
& + \lambda^j[2\partial_\tau\bar{\eta}_0\partial_j\eta^0 + (\partial_j\partial_\tau\bar{\eta}_0)\eta^0 + \partial_\tau\bar{\eta}_k\partial_j\eta^k + \partial_\tau\bar{\eta}_j\partial_k\eta^k + (\partial_k\partial_\tau\bar{\eta}_j)\eta^k] \\
& + \int \mathrm{d}^p\sigma'\{\bar{\eta}_0(\underline{\sigma}')[\{T_0, \chi^0(\underline{\sigma}')\}\eta^0 + \{T_j, \chi^0(\underline{\sigma}')\}\eta^j] \\
& + \bar{\eta}_j(\underline{\sigma}')[\{T_0, \chi^j(\underline{\sigma}')\}\eta^0 + \{T_k, \chi^j(\underline{\sigma}')\}\eta^k]\}\,, \\
L_{GF} = & \frac{1}{2\rho_{(0)}}(\partial_\tau\lambda^0 - \chi^0)(\partial_\tau\lambda_0 - \chi_0) + \frac{1}{2\rho_{(j)}}(\partial_\tau\lambda^j - \chi^j)(\partial_\tau\lambda_j - \chi_j)\,.
\end{aligned}$$

If one does not intend to pass to the Lagrangian formalism, one may restrict oneself to the minimal sector $\left(\Omega^{\min}, \chi^{\min}, H_\chi^{\min}\right)$. In particular, this means that Lagrange multipliers are not considered as dynamical variables anymore. With this particular gauge choice, $H_\chi^{\min}$ is a linear combination of the total constraints

$$H_\chi^{\min} = \int \mathrm{d}^p\sigma\left[\Lambda^0 T_0^{\mathrm{tot}}(\underline{\sigma}) + \Lambda^j T_j^{\mathrm{tot}}(\underline{\sigma})\right],$$

and we can treat here the Lagrange multipliers Λ^0, Λ^j as constants.

As a result, we have the possibility to quantize this dynamical system living in curved background.

2.3. Solving the Equations of Motion

The brane equations of motion and constraints in curved space-time are highly nonlinear and, *in general*, non exactly solvable. Different methods have been applied to solve them approximately or, if possible, exactly in a fixed background. On the other hand, quite general exact solutions can be found by using an appropriate ansatz, which exploits the symmetries of the underlying curved space-time [1-3]. We will use namely this approach.

From now on, we will work in the gauge $\lambda^m = \mathrm{const}$, in which the equations of motion may be written in the form

$$g_{LN}\left(\partial_\tau - \lambda^j\partial_j\right)^2 X^N + \Gamma_{L,MN}\left(\partial_\tau - \lambda^j\partial_j\right)X^M\left(\partial_\tau - \lambda^k\partial_k\right)X^N = 0\,.$$

First of all, we will look for background independent solution of these equations and of the constraints

$$g_{MN}\left(\partial_\tau - \lambda^j\partial_j\right)X^M\left(\partial_\tau - \lambda^k\partial_k\right)X^N = 0\,, \tag{9}$$

$$g_{MN}\left(\partial_\tau - \lambda^k\partial_k\right)X^M\partial_j X^N = 0\,. \tag{10}$$

It is easy to check that the solution is [1, 2]

$$X^M(\xi) = F^M(\lambda^i\xi^0 + \xi^i) \equiv F^M(\lambda^i\tau + \sigma^i)\,,$$

where F^M are D arbitrary functions of their arguments. The next step is to use the existing symmetries of the background metric. To this end, let us split the index $M = (\mu, a)$, $\{\mu\} \neq \{\emptyset\}$ and let us suppose that there exist a number of independent Killing vectors η_μ. Then in appropriate coordinates $\eta_\mu = \partial/\partial x^\mu$ and the metric does not depend on X^μ. Now our aim is to find an ansatz for X^M, which will allow us to separate the variables ξ^0 and ξ^i on the one hand, and on the other hand, to find the first integrals of a part of the equations of motion, corresponding to the symmetry of the curved background space-time. It turns out that the appropriate ansatz is:

$$X^\mu\left(\tau,\sigma^i\right) = C^\mu F(\lambda^i\tau + \sigma^i) + y^\mu(\tau), \quad C^\mu = \text{const}\,,$$
$$X^a\left(\tau,\sigma^i\right) = y^a(\tau)\,.$$

Inserting it in the equations of motion and constraints, one obtains the conserved quantities:

$$g_{\mu\nu}\dot{y}^\nu + g_{\mu a}\dot{y}^a = A_\mu = \text{const}\ .$$

The constraints (10) are identically satisfied when $A_\mu C^\mu = 0$. The remaining equations and the constraint (9) are reduced to

$$g_{aN}\ddot{y}^N + \Gamma_{a,MN}\dot{y}^M\dot{y}^N = 0$$
$$g_{MN}\dot{y}^M\dot{y}^N = 0\,.$$

Using the obtained first integrals, these equalities can be rewritten as

$$2\frac{\mathrm{d}}{\mathrm{d}\tau}\left(h_{ab}\dot{y}^b\right) - \left(\partial_a h_{bc}\right)\dot{y}^b\dot{y}^c + \partial_a V = 4\partial_{[a}A_{b]}\dot{y}^b$$
$$h_{ab}\dot{y}^a\dot{y}^b + V = 0\,, \tag{11}$$

where

$$h_{ab} \equiv g_{ab} - g_{a\mu}k^{\mu\nu}g_{\nu b}, \qquad V \equiv A_\mu A_\nu k^{\mu\nu} \quad A_a \equiv g_{a\mu}k^{\mu\nu}A_\nu\,,$$

and $k^{\mu\nu}$ is by definition the inverse of $g_{\mu\nu}$: $k^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu$. Thus, using the existence of an abelian isometry group G generated by the Killing vectors $\partial/\partial x^\mu$, the problem of solving the equations of motion and $p+1$ constraints in D-dimensional curved space-time $\mathcal{M}_D$ with metric g_{MN} is reduced to considering equations of motion and one constraint in the coset $\mathcal{M}_D/G$ with metric h_{ab}. As might be expected, an interaction with an effective gauge field appears in the Euler–Lagrange equations. In this connection, let us note that if we write down A_a as

$$A_a = A_a^\nu A_\nu ,$$

this establishes a correspondence with the usual Kaluza–Klein type notations and

$$g_{MN}\,\mathrm{d}y^M\,\mathrm{d}y^N = h_{ab}dy^a\,\mathrm{d}y^b + g_{\mu\nu}\left(\mathrm{d}y^\mu + A_a^\mu\,\mathrm{d}y^a\right)\left(\mathrm{d}y^\nu + A_b^\nu\,\mathrm{d}y^b\right) .$$

At this stage, we restrict the metric h_{ab} to be a diagonal one, i. e.

$$g_{ab} = g_{a\mu}k^{\mu\nu}g_{\nu b} \ \text{ for } \ a \neq b \,. \tag{12}$$

This allows us to transform further the equations for y^a and obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(h_{aa}\dot{y}^a\right)^2 &+ \dot{y}^a\partial_a\left(h_{aa}V\right) \\ &+ \dot{y}^a\sum_{b\neq a}\left[\partial_a\left(\frac{h_{aa}}{h_{bb}}\right)\left(h_{bb}\dot{y}^b\right)^2 - 4\partial_{[a}A_{b]}h_{aa}\dot{y}^b\right] = 0\,.\end{aligned} \tag{13}$$

To reduce the order of the differential equations (13) by one, we first split the index a in such a way that y^r is one of the coordinates y^a, and y^α are the others. Then we impose the sufficient conditions

$$\begin{aligned}\partial_\alpha\left(\frac{h_{\alpha\alpha}}{h_{aa}}\right) = 0\,, \qquad & \partial_\alpha\left(h_{rr}\dot{y}^r\right)^2 = 0\,, \\ \partial_r\left(h_{\alpha\alpha}\dot{y}^\alpha\right)^2 = 0\,, \qquad & A_\alpha = \partial_\alpha f\,.\end{aligned} \tag{14}$$

The result of integrations, compatible with (11) and (12), is the following

$$\begin{aligned}\left(h_{\alpha\alpha}\dot{y}^\alpha\right)^2 &= D_\alpha\left(y^a \neq y^\alpha\right) + h_{\alpha\alpha}\left[2\left(A_r - \partial_r f\right)\dot{y}^r - V\right] = E_\alpha(y^\beta)\,, \\ \left(h_{rr}\dot{z}^r\right)^2 &= h_{rr}\left[(S_\alpha - 1)V - \sum_\alpha \frac{D_\alpha}{h_{\alpha\alpha}}\right] + \left[S_\alpha(A_r - \partial_r f)\right]^2 = E_r(y^r)\,,\end{aligned} \tag{15}$$

where D_α, E_α, E_r are arbitrary functions of their arguments

$$\dot{z}^r \equiv \dot{y}^r + \frac{S_\alpha}{h_{rr}}\left(A_r - \partial_r f\right),$$

and S_α is the number of the coordinates y^α. To find solutions of the above equations without choosing particular metric, we have to fix all coordinates y^a except one. If we denote it by y^A, then the *exact* solutions of the equations of motion and constraints for a null p-brane in the considered curved background are given by

$$X^{\mu}\left(X^{A},\sigma^{j}\right)=X_{0}^{\mu}+C^{\mu}F\left(\lambda^{j}\tau+\sigma^{j}\right)$$
$$-\int_{X_0^A}^{X^A}k_{0}^{\mu\nu}\left[g_{\nu A}^{0}\mp A_{\nu}\left(-\frac{h_{AA}^{0}}{V^{0}}\right)^{1/2}\right]\mathrm{d}u\,,$$

$$\tau\left(X^{A}\right)=\tau_{0}\pm\int_{X_0^A}^{X^A}\left(-\frac{h_{AA}^{0}}{V^{0}}\right)^{1/2}\mathrm{d}u\,.$$

3. Exact Solutions for the Tensile 1-brane

To begin with, we write down the bosonic string action in D-dimensional curved space-time $\mathcal{M}_D$ with metric tensor g_{MN}

$$S=\int \mathrm{d}^{2}\xi\mathcal{L}\,,\qquad \mathcal{L}=-\frac{T}{2}\sqrt{-\gamma}\gamma^{mn}\partial_{m}X^{M}\partial_{n}X^{N}g_{MN}(X)\,,$$

where, as usual, T is the string tension and γ is the determinant of the auxiliary metric γ_{mn}.

Here we would like to consider tensile and null (tensionless) strings on equal footing, so we have to rewrite the action in a form in which the limit $T\rightarrow 0$ could be taken. To this end, we set [4]

$$\gamma^{mn}=\begin{pmatrix}-1 & \lambda^{1}\\ \lambda^{1} & \left(2\lambda^{0}T\right)^{2}-\left(\lambda^{1}\right)^{2}\end{pmatrix}$$

and obtain

$$\mathcal{L}=\frac{1}{4\lambda^{0}}g_{MN}(X)\left(\partial_{0}-\lambda^{1}\partial_{1}\right)X^{M}\left(\partial_{0}-\lambda^{1}\partial_{1}\right)X^{N}$$
$$-\lambda^{0}T^{2}g_{MN}(X)\partial_{1}X^{M}\partial_{1}X^{N}\,.$$

The equations of motion and constraints following from this Lagrangian density are ($\lambda^{m}=\mathrm{const}$):

$$g_{LN}\left(\partial_{0}-\lambda^{1}\partial_{1}\right)^{2}X^{N}+\Gamma_{L,MN}\left(\partial_{0}-\lambda^{1}\partial_{1}\right)X^{M}\left(\partial_{0}-\lambda^{1}\partial_{1}\right)X^{N}$$
$$=\left(2\lambda^{0}T\right)^{2}\left(\partial_{1}^{2}X^{K}+\Gamma_{MN}^{K}\partial_{1}X^{M}\partial_{1}X^{N}\right)\,,$$

$$g_{MN}(X)\left(\partial_{0}-\lambda^{1}\partial_{1}\right)X^{M}\left(\partial_{0}-\lambda^{1}\partial_{1}\right)X^{N}+\left(2\lambda^{0}T\right)^{2}g_{MN}(X)\partial_{1}X^{M}\partial_{1}X^{N}$$
$$=0\,,$$

$$g_{MN}(X)\left(\partial_{0}-\lambda^{1}\partial_{1}\right)X^{M}\partial_{1}X^{N}=0\,.$$

The background independent solution of the equations of motion (but not of the constraints) is [4]

$$X^M(\tau,\sigma) = F^M_\pm[w_\pm(\tau,\sigma)]\,, \qquad w_\pm(\tau,\sigma) = (\lambda^1 \pm 2\lambda^0 T)\tau + \sigma\,.$$

The ansatz with the searched properties is given by

$$\begin{aligned} X^\mu(\tau,\sigma) &= C^\mu_\pm w_\pm + y^\mu(\tau), \qquad C^\mu_\pm = \text{const}\,,\\ X^a(\tau,\sigma) &= y^a(\tau)\,. \end{aligned}$$

Applying this ansatz for the equations of motion and constraints one obtains

$$\begin{gathered} g_{KL}\ddot{y}^L + \Gamma_{K,MN}\dot{y}^M\dot{y}^N \pm 4\lambda^0 T C^\mu_\pm \Gamma_{K,\mu N}\dot{y}^N = 0\,,\\ g_{MN}(y^a)\dot{y}^M\dot{y}^N \pm 2\lambda^0 T C^\mu_\pm \left[g_{\mu N}(y^a)\dot{y}^N \pm 2\lambda^0 T C^\nu_\pm g_{\mu\nu}(y^a)\right] = 0\,,\\ C^\mu_\pm \left[g_{\mu N}(y^a)\dot{y}^N \pm 2\lambda^0 T C^\nu_\pm g_{\mu\nu}(y^a)\right] = 0\,. \end{gathered}$$

Obviously, this system of two constraints is equivalent to the following one

$$\begin{gathered} g_{MN}(y^a)\dot{y}^M\dot{y}^N = 0\,,\\ C^\mu_\pm \left[g_{\mu N}(y^a)\dot{y}^N \pm 2\lambda^0 T C^\nu_\pm g_{\mu\nu}(y^a)\right] = 0\,, \end{gathered}$$

which we will use from now on.
The integration of a part of the Euler–Lagrange equations leads to the conserved quantities [4]

$$g_{\mu\nu}\dot{y}^\nu + g_{\mu a}\dot{y}^a \pm 2\lambda^0 T C^\nu_\pm g_{\mu\nu} = A^\pm_\mu = \text{const}\,, \qquad A^\pm_\mu C^\mu_\pm = 0\,.$$

Using the obtained first integrals, one reduces the problem to solving the equations

$$\begin{gathered} 2\frac{\mathrm{d}}{\mathrm{d}\tau}\left(h_{ab}\dot{y}^b\right) - (\partial_a h_{bc})\dot{y}^b\dot{y}^c + \partial_a V^\pm = 4\partial_{[a}A^\pm_{b]}\dot{y}^b\\ h_{ab}\dot{y}^a\dot{y}^b + V^\pm = 0\,, \end{gathered}$$

where

$$V^\pm \equiv A^\pm_\mu A^\pm_\nu k^{\mu\nu} + \left(2\lambda^0 T\right)^2 C^\mu_\pm C^\nu_\pm g_{\mu\nu}\,, \qquad A^\pm_a \equiv g_{a\mu}k^{\mu\nu}A^\pm_\nu\,.$$

Obviously, these equalities have the same form as before with the replacements $V \to V^\pm$, $A_a \to A^\pm_a$. Therefore, we can write down the exact solution in this case right now, and it is (under the same conditions on the metric):

$$X^{\mu}\left(X^{A},\sigma\right) = X_0^{\mu} + C_{\pm}^{\mu}\left(\lambda^1\tau+\sigma\right) - \int_{X_0^A}^{X^A} k_0^{\mu\nu}\left[g_{\nu A}^0 \mp A_{\nu}^{\pm}\left(-\frac{h_{AA}^0}{V^{\pm 0}}\right)^{1/2}\right] \mathrm{d}u\,,$$

$$\tau\left(X^A\right) = \tau_0 \pm \int_{X_0^A}^{X^A}\left(-\frac{h_{AA}^0}{V^{\pm 0}}\right)^{1/2} \mathrm{d}u\,.$$

4. Explicit Examples

First of all, let us write down the generic structure of the solutions as functions of the coordinate X^A. For tensionless p-branes, i. e. $T=0$, p — arbitrary, it is

$$X^{\mu}\left(X^A,\sigma^j\right) = X_0^{\mu} + C^{\mu}F\left(\lambda^j\tau+\sigma^j\right) \pm \lim_{T\to 0} I_{\pm}^{\mu}\left(X^A;T\right)$$
$$\tau\left(X^A\right) = \tau_0 \pm \lim_{T\to 0} I_0^{\pm}\left(X^A;T\right)\,.$$

For tensile strings, i. e. $T \neq 0,\ p=1$, we have

$$X^{\mu}\left(X^A,\sigma\right) = X_0^{\mu} + C_{\pm}^{\mu}\left(\lambda^1\tau+\sigma\right) \pm I_{\pm}^{\mu}\left(X^A;T\right)$$
$$\tau\left(X^A\right) = \tau_0 \pm I_0^{\pm}\left(X^A;T\right)\,.$$

In our examples below, we will give the expressions for $I_{\pm}^{\mu}$ and $I_0^{\pm}$.
Let us first consider an exact null p-brane solution for a four dimensional cosmological **Kasner background**. Namely, the line element is ($x^0 \equiv t$)

$$\mathrm{d}s^2 = g_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N = -(\mathrm{d}t)^2 + \sum_{\mu=1}^{3} t^{2q_{\mu}}(\mathrm{d}x^{\mu})^2\,,$$
$$\sum_{\mu=1}^{3} q_{\mu} = 1, \qquad \sum_{\mu=1}^{3} q_{\mu}^2 = 1\,. \tag{16}$$

Without using the Kasner constraints (16), the solution is given by [4]

$$I^{\mu}(t) = \mathrm{const} - \frac{\sqrt{\pi}}{A_2^{\pm}}\sum_{k=0}^{\infty}\frac{\left(A_3^{\pm}/A_2^{\pm}\right)^{2k}}{k!\Gamma\left(1/2-k\right)}\frac{t^{\mathcal{P}}}{\mathcal{P}}\,{}_2F_1\left(\frac{1}{2}+k,\,\frac{\mathcal{P}}{2(q_2-q_1)};\right.$$
$$\left.\frac{2(q_2-q_3)k+3q_2-2q_1+1-2q_{\mu}}{2(q_2-q_1)};-\frac{A_1^{\pm 2}}{A_2^{\pm 2}}t^{2(q_2-q_1)}\right),$$
$$\mathcal{P} \equiv 2(q_2-q_3)k+q_2+1-2q_{\mu}, \qquad q_{\mu}=(q_1,q_2,q_3)\,,\quad \text{for } q_1>q_2\,,$$

and

$$I^{\mu}(t) = \mathrm{const} + \frac{\sqrt{\pi}}{A_1^{\mp}} \sum_{k=0}^{\infty} \frac{\left(A_3^{\pm}/A_1^{\mp}\right)^{2k}}{k!\Gamma\left(1/2-k\right)} \frac{t^{\mathcal{Q}}}{\mathcal{Q}} {}_2F_1\Bigg(\frac{1}{2}+k, \frac{\mathcal{Q}}{2(q_1-q_2)};$$
$$\frac{2(q_1-q_3)k+3q_1-2q_2+1-2q_\mu}{2(q_1-q_2)}; -\left(\frac{A_2^{\pm}}{A_1^{\mp}}\right)^2 t^{2(q_1-q_2)}\Bigg),$$
$$\mathcal{Q} \equiv 2(q_1-q_3)k + q_1 + 1 - 2q_\mu\,, \quad \text{for } q_1 < q_2\,,$$

where ${}_2F_1(a,b;c;z)$ is the Gauss hypergeometric function and $\Gamma(z)$ is the Euler's Γ-function. The expressions for $I_0^{\pm}$ are obtainable from the above ones by setting $q_\mu = 0$. Because there are no restrictions on q_μ, except $q_1 \neq q_2 \neq q_3$, this probe brane solution is also valid in generalized Kasner type backgrounds arising in superstring cosmology [5]. In string frame, the effective Kasner constraints for the four dimensional dilaton-moduli-vacuum solution are

$$\sum_{\mu=1}^{3} q_\mu = 1 + \mathcal{K}\,, \qquad \sum_{\mu=1}^{3} q_\mu^2 = 1 - \mathcal{B}^2\,,$$
$$-1 - \sqrt{3\left(1-\mathcal{B}^2\right)} \leq \mathcal{K} \leq -1 + \sqrt{3\left(1-\mathcal{B}^2\right)}\,, \qquad \mathcal{B}^2 \in [0,1]\,.$$

In Einstein frame, the metric has the same form, but in new, rescaled coordinates and with new powers $\tilde{q}_\mu$ of the scale factors. The generalized Kasner constraints are also modified as follows

$$\sum_{\mu=1}^{3} \tilde{q}_\mu = 1\,, \qquad \sum_{\mu=1}^{3} \tilde{q}_\mu^2 = 1 - \tilde{\mathcal{B}}^2 - \frac{1}{2}\tilde{\mathcal{K}}^2\,, \qquad \tilde{\mathcal{B}}^2 + \frac{1}{2}\tilde{\mathcal{K}}^2 \in [0,1]\,.$$

Actually, the obtained tensionless p-brane solution is also relevant to considerations within a **pre-big bang** context, because there exist a class of models for pre-big bang cosmology, which is a particular case of the given generalized Kasner backgrounds [5].

Our next example is for a tensile string, evolving in the **Kerr space–time** with metric taken in the form

$$g_{00} = -\left(1 - \frac{2Mr}{\rho^2}\right), \qquad g_{11} = \frac{\rho^2}{\Delta}\,, \qquad g_{22} = \rho^2\,,$$
$$g_{33} = \left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\rho^2}\right)\sin^2\theta\,, \qquad g_{03} = -\frac{2Mar\sin^2\theta}{\rho^2}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta\,, \qquad \Delta = r^2 - 2Mr + a^2\,,$$

M is the mass and a is the angular momentum per unit mass of the Kerr black hole. Now the metric does not depend on x^0 and x^3, so that $\mu = 0, 3$, $a = 1, 2$. The corresponding exact string solution, as a function of the radial coordinate r, is described by the integrals ($\theta = \theta_0 = \mathrm{const}$):

$$I^0_{\pm}(r;T) = \int_{r_0}^{r} \mathrm{d}r \Bigg[A_0^{\pm} a^2 \Delta \sin^2\theta_0 - 2A_3^{\pm} Mar - A_0^{\pm}\left(r^2+a^2\right)^2 \Bigg] \left[-\Delta^3 D_2(r;T)\right]^{-1/2},$$

$$I^3_{\pm}(r;T) = \int_{r_0}^{r} \mathrm{d}r \left(\frac{A_3^{\pm}\Delta}{\sin^2\theta_0} - 2A_0^{\pm} Mar - A_3^{\pm} a^2 \right) \left[-\Delta^3 D_2(r;T)\right]^{-1/2},$$

$$I_0^{\pm}(r;T) = \int_{r_0}^{r} \mathrm{d}r \rho_0^2 [-\Delta D_2(r;T)]^{-1/2}, \quad \rho_0^2 = r^2 + a^2\cos^2\theta_0,$$

$$\begin{aligned} D_2(r;T) = &-\frac{1}{\Delta}\left[\left(A_0^{\pm}\right)^2 \left(r^2+a^2\right)^2 + 4A_0^{\pm}A_3^{\pm}Mar + \left(A_3^{\pm}\right)^2 a^2 \right] \\ &+ \left(A_0^{\pm}\right)^2 a^2 \sin^2\theta_0 + \frac{\left(A_3^{\pm}\right)^2}{\sin^2\theta_0} + \left(2\lambda^0 T\right)^2 \left(C_{\pm}^0\right)^2 \left(a^2\sin^2\theta_0 - \Delta\right) \\ &+ \left(2\lambda^0 T\right)^2 \sin^2\theta_0 \Bigg[\left(C_{\pm}^3\right)^2 \left(r^2+a^2\right)^2 - 4C_{\pm}^0 C_{\pm}^3 Mar \\ &- \left(C_{\pm}^3\right)^2 a^2 \Delta \sin^2\theta_0 \Bigg]. \end{aligned}$$

Analogously, one can write down the solution as a function of θ when the radial coordinate is kept fixed. Moreover, in the zero tension limit, one can find the orbit $r = r(\theta)$, which is given by:

$$\int_{r_0}^{r} \frac{\mathrm{d}r}{\Delta(r) E_1^{1/2}(r)} = \pm \int_{\theta_0}^{\theta} \frac{\mathrm{d}\theta}{E_2^{1/2}(\theta)}$$

where

$$E_1(r) = \frac{1}{\Delta^2}\left[A_0^2\left(r^2+a^2\right)^2 + 4A_0A_3Mar + A_3^2 a^2 \right] - \frac{d}{\Delta},$$

$$E_2(\theta) = d - \left(A_0^2 a^2 \sin^2\theta + \frac{A_3^2}{\sin^2\theta} \right), \quad d = \mathrm{const}.$$

The possibility to obtain this result is due to the fact that on the one hand the conditions (14) are satisfied, and on the other hand that in (15) the variables r

and θ can be separated. This corresponds to the separation of the variables in the Hamilton–Jacobi equation for the Kerr metric, connected to the existence of a nontrivial Killing tensor of second rank for this space–time.

5. Concluding Remarks

In this talk we gave a short review of the results on the p-brane dynamics in curved backgrounds, received in [1, 4]. The new point here is the generalization of the previously obtained null p-brane solutions to the case of more general class of background metrics. The tensile 1-brane exact solution in Kerr space–time is also new one.

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