Third International Conference on
Geometry, Integrability and Quantization
June 14–23, 2001, Varna, Bulgaria
Ivaïlo M. Mladenov and Gregory L. Naber, Editors
Coral Press, Sofia 2001, pp 105–140

INVARIANTS OF SMOOTH FOUR-MANIFOLDS: TOPOLOGY, GEOMETRY, PHYSICS

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Abstract. The profound and beautiful interaction between smooth four-manifold topology and the quantum theory of fields often seems as impenetrable as it is impressive. The objective of this series of lectures is to provide a very modest introduction to this interaction by describing, in terms as elementary as possible, how Atiyah and Jeffrey [1] came to view the partition function of Witten's first topological quantum field theory [21], which coincides with the zero-dimensional Donaldson invariant, as an "Euler characteristic" for an infinite-dimensional vector bundle.

1. Motivation: Donaldson–Witten Theory

From 1982 to 1994 the study of smooth four-manifolds was dominated by the ideas of Simon Donaldson who showed how to construct remarkably sensitive differential topological invariants for such a manifold B from moduli spaces of anti-self-dual connections on principal $\mathbb{SU}(2)$ or $\mathbb{SO}(3)$ bundles over B (we will describe the simplest of these in Section 4). In 1988, Edward Witten [21], prompted by Atiyah, constructed a quantum field theory in which the Donaldson invariants appeared as expectation values of certain observables. This came to be known as *Donaldson–Witten theory* and the action underlying it was of the form

$$\begin{aligned} S_{\mathrm{DW}} \propto \int_B \operatorname{Tr}\Big\{ &-\frac{1}{4}F_\omega \wedge *F_\omega - \frac{1}{4}F_\omega \wedge F_\omega - \frac{1}{2}\psi \wedge [\phi, \psi] \\ &+ \mathrm{i}\,\mathrm{d}^\omega \chi \wedge \psi + 2\mathrm{i}[\chi, *\chi]\lambda - \mathrm{i} * (\phi \Delta_0^\omega \lambda) + \chi \wedge * \mathrm{d}^\omega \eta \Big\}, \end{aligned} \tag{1.1}$$