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MODERN APPROACHES TO THE QUANTIZATION OF GAUGE THEORIES

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Abstract. In the Batalin–Vilkovisky field-antifield formalism a classical mechanical system is described by a solution of the classical master equation. The quantization of this general gauge theory in the Lagrangian approach can be accomplished in closed form. The AKSZ-formalism is a geometrical construction of such a solution as a QP-manifold. This can be extended and applied to topological quantum field theories.

1. Introduction

In physics the fundamental interactions are governed by gauge theories. One usually does not want to eliminate the gauge degrees of freedom, because they ensure manifest covariance, the locality of the interactions and they are convenient for calculational purposes. The quantization is not always straightforward. In general it involves the introduction of ghost fields. It is useful to introduce these ghosts at the classical level, then one is able to quantize the theory in a canonical way, since all necessary parameters are involved from the beginning. These so-called **pseudo-classical theories** are formulated by the use of fermionic degrees of freedom, which lead to Grassmann algebras and "supersymmetric theories".

Ghosts appeared in physics for the first time in the Faddeev–Popov quantization procedure [8]. This relies on the path integral quantization; the ghosts are introduced by dividing out the volume of gauge transformations in function space, which leads to a finite path integral measure. One is left with a summation over the equivalence classes of gauge fields, bearing in mind the gauge invariance, so as to avoid double counting. The presence of ghosts can be here understood as a measure effect.

Later on it was realized that the action retains a nilpotent, odd, global symmetry involving transformations of fields and ghosts. This Becci–Rouet–Stora–Tyutin (BRST) symmetry [5, 19] is what remains from the original gauge invariance. The BRST symmetry mixes the ghosts with the other fields, all fields including the ghosts are regarded as different components of a single geometrical object. The classical phase space is extended by introducing Grassmann valued 'coordinates'. A generalized Poisson bracket, which induces a closed 2-form, is defined on this space. The geometry of this phase space is called supersymplectic geometry. The central idea is to substitute the original gauge invariance by a rigid fermionic symmetry Ω, acting on the extended phase space, which contains the new variables, called ghosts. The key property of the BRST operator, $\delta_\Omega^2 = 0$, is that it is a nilpotent derivation, so one can construct **cohomology groups** $\mathbb{H}^k(\delta_\Omega)$. The original gauge invariance is recovered when one passes to cohomology, the BRST symmetry completely captures it and leads to a simpler formulation of the theory. The BRST construction works well in the Hamilton formalism.

A Lagrange formulation of the theory has the advantage of manifest covariance, because the Lagrangian L is a scalar under space-time transformations. This is better for a theory with local symmetries. The construction of a BRST invariant Lagrangian L_{eff}, with the requirement that the ghost number of L_{eff} has to be zero, leads to the introduction of variables with negative ghost number. Hence the configuration space has to be extended again with the so-called antighosts. This procedure was initiated by the work of Zinn–Justin [21], and Batalin and Vilkovisky [2], who introduced an effective action on a doubled configuration space and defined an antibracket on it. The BRST invariance of the action can expressed in the master equation $(S, S) = 0$, the solutions generate the full dynamics of the theory.

In the formalism of Alexandrov, Kontsevich, Schwarz and Zaboronsky (AKSZ) the solutions of the master equation which are of physical interest are directly constructed in a geometrical way [1].

2. General Gauge Theory

2.1. Canonical Formalism

The non-abelian Yang–Mills theory is the most familiar example of a gauge structure. In this case, when a choice of a basis is made, the structure constants of the underlying Lie group determine the commutator algebra. The Jacobi identity, which expresses the associativity of the Lie group, must be satisfied.

In other types of theories, which are called reducible, the generators of the gauge transformations are not independent. There exists a 'residual gauge invariance' for gauge transformations, this makes the structure of the theory more complicated than in the Yang–Mills case. Another complication occurs when the commutator of two gauge transformations produces a term which vanishes "on-shell", i. e. when equations of motion appear in the gauge algebra.

Now we will recall the canonical formalism in a compact notation. Consider a system whose dynamics is governed by a classical action $S_0[\phi]$, depending on n different fields ϕ^i with $i = 1, \ldots, n = n_+ + n_-$, where n_+ is the number of bosons and n_- is the number of fermions. In general i can label space-time indices of tensor fields, spinor indices of fermion fields or distinguish between different types of generic fields. Let $\epsilon(\phi^i) = \epsilon_i$ denote the statistical parity, i. e. the Grassmannn parity of ϕ^i. Each ϕ^i is either a commuting bosonic field with parity $\epsilon_i = 0$ or an anticommuting fermionic field with $\epsilon_i = 1$, so one has $\phi^i(x)\phi^j(y) = (-1)^{\epsilon_i\epsilon_j}\phi^j(y)\phi^i(x)$ according to the Koszul sign rule. Here the new variables are introduced at the classical level.

Assume that the action $S_0[\phi]$ is invariant under a set of m_0, $m_0 \leq n$, non-trivial gauge transformations, which read in infinitesimal form

$$\delta\phi^i = R^i_\alpha \varepsilon^\alpha, \quad \text{where} \quad \alpha = 1, 2, \ldots, m_0 .$$

This is the compact notation, where

ε^α is an infinitesimal gauge parameter

with parity $\epsilon_\alpha = \begin{cases} 0 & \text{ordinary symmetry} \\ 1 & \text{supersymmetry} \end{cases}$

R^i_α are generators of gauge transformations

with parity $\epsilon(R^i_\alpha) = (\epsilon_i + \epsilon_\alpha) \bmod 2$.

Later on the gauge parameters will be turned into ghosts. Let $S_{0,i}[\phi]$ denote the variation of the action with respect to ϕ^i

$$S_{0,i}[\phi] \equiv \left. \frac{\partial_r S}{\partial \phi^i} \right|_{\phi_0}$$

where the index r denotes the right derivative. The distinction between left and right derivatives is necessary in the context of Grassmann algebras with fermionic and bosonic variables. The statement that the action is invariant under gauge transformations of the form $\delta\phi^i = R^i_\alpha \varepsilon^\alpha$ means that the Noether identities hold

$$S_{0,i} R^i_\alpha = 0 .$$

A theory is irreducible when all gauge transformations are independent, and L-stage reducible, when a dependence exists. The Noether identities can be thought of as a definition of when a theory is invariant under gauge transformations.

To commence canonical quantization of a theory, one searches for solutions of the classical equations of motion $S_{0,i} = 0$ and then expands about these solutions. Assume there exists at least one stationary point ϕ_0 so that

$$S_{0,i}|_{\phi_0} = 0\,.$$

This equation defines a surface Σ in configuration space, the restriction of the full space to the physical hypersurface. As a consequence of the Noether identities it is necessary to assume certain **regularity conditions** on Σ. The key consequence of the regularity conditions for an arbritrary functional F of the fields is

$$F(\phi)|_\Sigma = 0 \quad\Longrightarrow\quad F(\Sigma) = S_{0,i}\lambda^i(\phi)$$

i. e. if F vanishes "on-shell" then F must be a linear combination of the equations of motion. This leads to a generalization of the Noether identities

$$S_{0,i}\lambda^i = 0 \quad \text{with} \quad \lambda^i = R^i_{0\alpha_0}\lambda^{\alpha_0} + S_{0,j}T^{ij}$$

where the last term denotes a trivial gauge transformation. For an irreducible theory the generators $R^i_{0\alpha_0}\lambda^{\alpha_0}$ are independent "on-shell":

$$\operatorname{rank} R^i_{0\alpha_0}\lambda^{\alpha_0}|_\Sigma = m_0$$

where m_0 is the number of gauge transformations, so at the quantum level m_0 ghosts will be needed. It is useful to introduce these ghosts at the classical level. The rank of the Hessian is

$$\operatorname{rank} \frac{\partial_l\partial_r S}{\partial\phi^j\partial\phi^i}|_\Sigma = n - m_o = n_{\text{dof}}$$

where n is the number of fields, and n_{dof} the net number of the degrees of freedom that enter dynamically in S_0, regardless of whether or not they propagate.

For the reducible case the generators are not independent

$$\operatorname{rank} R^i_{0\alpha_0}\lambda^{\alpha_0}|_\Sigma \leq m_0\,.$$

2.2. BRST Formalism

Variational principles lead to the classical Poisson bracket on the phase space $\{q^i; p_i\}$. Gauge transformations G are local symmetry transformations $\delta q^i, \delta p_i$; they leave the Hamiltonian invariant

$$\delta H = \{H, G\} = 0.$$

The original phase space must be extended by one degree of freedom for each symmetry transformation. In the BRST formalism the solutions of the constraints will be identified with the cohomology classes of a nilpotent operator Ω. The construction of Ω and the extension of the phase space will be done with ghosts. Define the generalized Poisson bracket

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} + (-1)^{\epsilon_F}\frac{\partial_l F}{\partial \theta^\alpha}\frac{\partial_l G}{\partial \pi_\alpha} - \frac{\partial_l F}{\partial \pi_\alpha}\frac{\partial_l G}{\partial \theta^\alpha}$$

on the extended phase space $\{q^i;\, p_i;\, \theta^\alpha;\, \pi_\alpha\}$, where q^i are the coordinates, p_i their conjugate momenta, and θ^α the ghosts with their conjugate momenta π_α. Let F and G are some functions on this space. The BRST operator Ω generates ghost dependent symmetry transformations of the classical phase variables $\delta_\Omega q^i$, $\delta_\Omega p_i$. Now define BRST transformations for the ghosts and require nilpotence, $\delta_\Omega^2 = 0$.

Let F be a gauge invariant physical quantity. F has to be BRST invariant

$$\delta_\Omega F = -\{\Omega, F\} = 0$$

and is said to be BRST-closed. The non-trivial solutions

$$F_0 = \delta_\Omega F_1 = -\{\Omega, F_1\}$$

are called BRST-exact. Hence F_0 depends on the ghosts, it is non-physical and must be divided out. This defines the BRST cohomology

$$\mathbb{H}(\delta_\Omega) = \frac{\operatorname{Ker}\delta_\Omega}{\operatorname{Im}\delta_\Omega}\,.$$

With the requirement of BRST invariance one can construct an effective Hamiltonian in the extented phase space

$$H_{\text{eff}} = H_0 - \delta_\Omega \psi$$

which consists of the classical part modulo a BRST-exact term. The latter acts only in the non-physical sector, ψ is a function with ghost number $N_g(\psi) = -1$, called a gauge fermion. It ensures a zero ghost number of the whole Hamiltonian. Here a variable with negative ghost number enters for the first time, from now on it will be called an antighost. The construction of an equivalent BRST invariant Lagranian is straightforward

$$L_{\text{eff}} = L_0 - \delta_\Omega \psi \,.$$

It is quite natural to construct with these variables an effective action on a doubled configuration space for fields

$$S_{\text{eff}}[\Phi^A; \Phi^*_A] = S_0 + \int \mathrm{d}t\, \delta_\Omega \Phi^A \Phi^*_A|_{\Phi^*_A = \frac{\partial \psi}{\partial \Phi^A}}$$

where $\Phi^*_A = \dfrac{\partial \psi}{\partial \Phi^A}$ is the restriction to the physical hypersurface. The 'antifields' are the sources of the BRST variations of the fields

$$\delta_\Omega \Phi^A = (-1)^{\epsilon_A} \frac{\delta S}{\delta \Phi^*_A} \,.$$

This leads to the Batalin–Vilkovisky field-antifield formalism, which is presented in the next section. For more details see [11, 15] and [10].

3. The Batalin–Vilkovisky Formalism

3.1. Fields and Antifields

Introduce a system

$$z^a = \{\Phi^A; \Phi^*_A\} \quad \text{with} \quad A = 1, \ldots, N \quad \text{and} \quad a = 1, \ldots, 2N$$

of fields Φ^A with Grassmann parity

$$\epsilon_A = \begin{cases} 0 & \text{boson} \\ 1 & \text{fermion} \end{cases}$$

and antifields Φ^*_A with opposite statistics $\epsilon^*_A = (\epsilon_A + 1) \bmod 2$, which carry ghost number

$$\mathrm{gh}[\Phi^A] \quad \text{and} \quad \mathrm{gh}[\Phi^*_A] = -\,\mathrm{gh}[\Phi^A] - 1 \,.$$

The collection of fields and antifields for a L-stage reducible theory is

$$\Phi^A = \{\phi^i; C_s^{\alpha_s}\} \quad \text{and} \quad \Phi^*_A = \{\phi^*_i; C^*_{s,\alpha_s}\}$$

with $s = 0, \ldots, L$ for L-stage reducible and $\alpha_s = 1, \ldots, m_s$ for s-level gauge invariances. For notational convenience define

$$C_{-1}^{\alpha_{-1}} \equiv \phi^i, \quad C^*_{-1,\alpha_{-1}} \equiv \phi^*_i \quad \text{with} \quad \alpha_{-1} = i \,.$$

In summary we have

fields	parity	ghostnumber	antifields	parity	ghostnumber
$C_{-1}^{\alpha_{-1}} = \phi^i$	ϵ_i	0	$C^*_{-1,\alpha_{-1}} = \phi^*_i$	ϵ^*_i	-1
$C_0^{\alpha_0} = C$	$\epsilon_{\alpha_0} + 1$	1	$C^*_{0,\alpha_0} = C^*$	$\epsilon^*_{\alpha_0}$	-2
$C_s^{\alpha_s}$	$\epsilon_{\alpha_s} + s + 1$ mod 2	$s+1$	C^*_{s,α_s}	$\epsilon^*_{\alpha_0} + s$ mod 2	$-s-2$

In the space of fields and antifields, one can define a bracket relation, the so-called antibracket or the Batalin–Vilkovisky bracket

$$(F,G) = \frac{\partial_r F}{\partial \Phi^A}\frac{\partial_l G}{\partial \Phi^*_A} - \frac{\partial_r F}{\partial \Phi^*_A}\frac{\partial_l G}{\partial \Phi^A} = \frac{\partial_r F}{\partial z^a}\xi^{ab}\frac{\partial_l G}{\partial z^b}$$

with

$$\xi^{ab} = \begin{pmatrix} 0 & \delta^A_B \\ -\delta^A_B & 0 \end{pmatrix}, \qquad \xi^{ab} = -\xi^{ba}, \qquad \epsilon\left(\xi^{ab}\frac{\partial}{\partial z^b}\right) = (\epsilon_a + 1) \bmod 2\,.$$

It is analogous to the generalized Poisson bracket with the replacement

$$\epsilon_F \longrightarrow \epsilon_{F+1}, \qquad \epsilon_G \longrightarrow \epsilon_{G+1}\,.$$

The antibracket carries the ghost number 1

$$\mathrm{gh}[(F,G)] = \mathrm{gh}(F) + \mathrm{gh}(G) + 1$$

and has odd statistics

$$\epsilon[(F,G)] = (\epsilon_F + \epsilon_G + 1) \bmod 2\,.$$

Therefore $(\cdot\,,\cdot)$ defines an odd symplectic structure. Its properties are

$$(F,F) = 0 \quad \text{for any fermion}$$
$$\text{and} \quad (B,B) = 2\frac{\partial_r B}{\partial \Phi^A}\frac{\partial_l B}{\partial \Phi^*_A} \neq 0 \quad \text{for any boson}$$

which is opposite to the expectation for the Poisson bracket. The functions on the space of fields and antifields form together with the Batalin–Vilkovisky bracket a graded algebra, as the generalized Poisson bracket does for functions of the phase space. These are special cases of a graded Lie algebra, which is defined in Section 3.3.

3.2. The Master Equation and its BRST Symmetry

Start with a functional $S[\Phi^A; \Phi^*_A]$, which has the dimension of an action, zero ghost number $\mathrm{gh}[S] = 0$ and even statistics $\epsilon_S = 0$. The classical master equation requires that the Batalin–Vilkovisky bracket of this bosonic functional vanishes,

$$(S, S) = 2\frac{\partial_r S}{\partial \Phi^A}\frac{\partial_l S}{\partial \Phi^*_A} = 0\,.$$

Not every solution of the master equation produces a dynamical system, only the proper solution is of interest. This kind of solution contains the original action and is obtained with the ghost number restriction and several boundary conditions, which guarantee the postulates of a gauge theory.
The proper solution reads

$$S[\Phi, \Phi^*] = S_0[\phi] + \sum_{s=0}^{L} C^*_{s-1,\alpha_{s-1}} R^{\alpha_{s-1}}_{s\alpha_s} C^{\alpha_s}_s + \cdots$$

with the boundary conditions $S[\Phi, \Phi^*]|_{\Phi^*=0} = S_0[\phi]$, which guarantee the classical limit and

$$\left.\frac{\partial_l \partial_r S}{\partial C^*_{s-1,\alpha_{s-1}}\partial C^{\alpha_s}_s}\right|_{\Phi^*=0} = R^{\alpha_{s-1}}_{s\alpha_s}(\phi)$$

which reflect the Noether identities. The master equation $(S, S) = 0$ is a very compact notation, with certain requirements it determines all gauge structure equations. The proper solution S is unique up to canonical transformations and the addition of trivial pairs.
The Batalin–Vilkovisky bracket between a field and an antifield is

$$(\Phi^A, \Phi^*_B) = \delta^A_B\,.$$

A canonical transformation for a field and an antifield is

$$\begin{aligned}\Phi^A &\to \bar{\Phi}^A = \Phi^A + \varepsilon(\Phi^A, F) + \mathcal{O}(\varepsilon^2)\\ \Phi^*_A &\to \bar{\Phi}^*_A = \Phi^*_A + \varepsilon(\Phi^*_A, F) + \mathcal{O}(\varepsilon^2)\end{aligned}$$

which preserves the antibracket up to the order of ε^2

$$(\bar{\Phi}^A, \bar{\Phi}^*_B) = \delta^A_B + \mathcal{O}(\varepsilon^2)\,.$$

The proper solution S has classical BRST-symmetry, which is a substitute for the gauge invariances.

$$\delta_S X \equiv (X, S) \quad \text{with} \quad X = X[\Phi, \Phi^*]$$

where the generator δ_S for the symmetry is S, the proper solution itself. The transformation rules are

$$\delta_S \Phi^A = \frac{\partial_l S}{\partial \Phi^*_A} \quad \text{and} \quad \delta_S \Phi^*_A = -\frac{\partial_l S}{\partial \Phi^A} = (-1)^{(\epsilon_A+1)} \frac{\partial_r S}{\partial \Phi^A} .$$

The symmetry of the action is guaranteed by the master equation

$$\delta_S S = 0 \quad \Longleftrightarrow \quad (S, S) = 0 .$$

δ_S is a nilpotent graded derivation

$$\begin{aligned} \delta_S^2 X &= 0 \\ \delta_S(XY) &= X(\delta_S Y) + (-1)^{\epsilon_Y}(\delta_S X)Y \end{aligned}$$

as follows from the properties of the antibracket.

3.3. Graded Algebras

A $\mathbb{Z}_2$-graded algebra is a vector space $\mathbb{L}$, which is in the simplest case the direct sum of

$$\mathbb{L} = \mathbb{L}_0 \oplus \mathbb{L}_1$$

together with a product $\circ$

$$u_i \circ u_j \in \mathbb{L}_{(i+j) \bmod 2} , \qquad u_i \in \mathbb{L}_i .$$

A product with these properties is called a grading. This concept can be extended to a $\mathbb{Z}_n$-graded algebra, which is a vector space

$$\mathbb{L} = \mathbb{L}_0 \oplus \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_{n-1}$$

with a product $u_i \circ u_j \in \mathbb{L}_{(i+j) \bmod n}$; and to a $\mathbb{Z}$-graded algebra, this is a vector space

$$\mathbb{L} = \mathbb{L}_0 \oplus \mathbb{L}_1 \oplus \cdots$$

with a product $u_i \circ u_j \in \mathbb{L}_{i+j+(\text{degree of } \circ)}$.

The $\mathbb{Z}_2$-graded algebra becomes a graded Lie algebra with the following requirements for the product $\circ$, now denoted by $[\cdot\,,\cdot]$:

$$\begin{array}{rrcl} \text{grading} & [x_i, x_j] & \in & \mathbb{L}_{(i+j) \bmod 2} \\ \text{(super)symmetry} & [x_i, x_j] & = & -(-1)^{ij}[x_j, x_i] \end{array}$$

and the Jacobi identity

$$(-1)^{km}[x_k, [x_l, x_m]] + (-1)^{lk}[x_l, [x_m, x_k]] + (-1)^{ml}[x_m, [x_k, x_l]] = 0 . \quad (1)$$

The signs change with the degree of the bracket. The formulas above are for an algebra with a Lie bracket of degree 0.

The most general algebra of this type is the Gerstenhaber algebra [9] which was originally defined for the cohomology groups of an associative ring. In general a Gerstenhabeer algebra consists of a $\mathbb{Z}$-graded vector space

$$\mathbb{V} = \mathbb{V}_0 \oplus \mathbb{V}_1 \oplus \cdots$$

together with a commutative associative multiplication of degree 0

$$\mathbb{V}_i \wedge \mathbb{V}_j \subseteq \mathbb{V}_{i+j}$$

and a graded Lie algebra structure of degree -1

$$[\mathbb{V}_i, \mathbb{V}_j]_{\mathcal{G}} \subseteq \mathbb{V}_{i+j-1}$$

i. e. for $v, w, y \in \mathbb{V}$, the Gerstenhaber bracket is graded antisymmetric

$$[v, w]_{\mathcal{G}} = -(-1)^{(\epsilon_v - 1)(\epsilon_w - 1)}[w, v]_{\mathcal{G}} \tag{2}$$

fulfils a graded Jacobi identity

$$\begin{aligned} [[v, w]_{\mathcal{G}}, y]_{\mathcal{G}} &+ (-1)^{(\epsilon_v - 1)(\epsilon_w + \epsilon_y)}[[w, y]_{\mathcal{G}}, v]_{\mathcal{G}} \\ &+ (-1)^{(\epsilon_y - 1)(\epsilon_v + \epsilon_w)}[[y, v]_{\mathcal{G}}, w]_{\mathcal{G}} = 0 \end{aligned} \tag{3}$$

and a graded Leibniz rule

$$[v, w \wedge y]_{\mathcal{G}} = [v, w]_{\mathcal{G}} \wedge y + (-1)^{(\epsilon_v - 1)\epsilon_w} w \wedge [v, y]_{\mathcal{G}}. \tag{4}$$

The Gerstenhaber bracket has the degree -1

$$\epsilon([v, w]_{\mathcal{G}}) = \epsilon_v + \epsilon_w - 1$$

so the main step was the redefinition of the Jacobi identity of a graded algebra where the bracket degree is 0, to one with $\epsilon([\cdot\,, \cdot]) \neq 0$. This was formulated by Gerstenhaber [9]:

Theorem 1. *The bracket $[\cdot\,, \cdot]_{\mathcal{G}}$ satisfies the super Jacobi identity if we declare elements of $M^k(\mathbb{V})$ to have degree $k - 1$, where $M^k(\mathbb{V})$ is the space of multilinear functions.*

The classical Poisson algebra can be generalized to a $\mathbb{Z}_2$-graded algebra by the extension of the phase space with fermionic degrees of freedom, the Grassmann variables.

The generalized Poisson bracket is defined on $\mathcal{F}(\{q; \theta\})$

$$\{F, G\} = \frac{\partial F}{\partial q^i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} + (-1)^{\epsilon_F}\frac{\partial_l F}{\partial \theta^\alpha}\frac{\partial_l G}{\partial \pi_\alpha} - \frac{\partial_l F}{\partial \pi_\alpha}\frac{\partial_l G}{\partial \theta^\alpha}\,. \tag{5}$$

This is graded antisymmetric in the classical supersymmetric sense

$$\{F, G\} = -(-1)^{\epsilon_F \epsilon_G} \{G, F\} \tag{6}$$

and fulfils the graded (super) Jacobi identity

$$\begin{aligned} \{\{F, G\}, H\} + (-1)^{\epsilon_F(\epsilon_G+\epsilon_H)} \{\{G, H\}, F\} \\ + (-1)^{\epsilon_H(\epsilon_F+\epsilon_G)} \{\{H, F\}, G\} = 0 \end{aligned} \tag{7}$$

and a graded (super) Leibniz rule

$$\{F, GH\} = \{F, G\}H + (-1)^{\epsilon_F \epsilon_G} G\{F, H\} . \tag{8}$$

The parity of the generalized Poisson bracket is 0, so this is the classical supersymmetric approach, which physicists are more familiar with

$$\epsilon[\{F, G\}] = \epsilon_F + \epsilon_G$$
$$\{B, B\} = 0 , \qquad \{F, F\} \neq 0 .$$

In the case of the Batalin–Vilkovisky algebra the phase space is extended again to a space with fields and antifields, both bosonic (even) and fermionic (odd). The degree of the bracket is $+1$, the grading of the algebra is $\mathbb{Z}_2$.
The Batalin–Vilkovisky bracket is defined on $\mathcal{F}(\{\Phi; \Phi^*\})$

$$(F, G) = \frac{\partial_r F}{\partial \Phi^A} \frac{\partial_l G}{\partial \Phi^*_A} - \frac{\partial_r F}{\partial \Phi^*_A} \frac{\partial_l G}{\partial \Phi^A} . \tag{9}$$

It is graded antisymmetric

$$(F, G) = -(-1)^{(\epsilon_F+1)(\epsilon_G+1)} (G, F) \tag{10}$$

and fulfils the graded Jacobi identity

$$\begin{aligned} ((F, G), H) + (-1)^{(\epsilon_F+1)(\epsilon_G+\epsilon_H)} ((G, H), F) \\ + (-1)^{(\epsilon_H+1)(\epsilon_F+\epsilon_G)} ((H, F), G) = 0 \end{aligned} \tag{11}$$

and the graded Leibniz rule

$$(F, GH) = (F, G)H + (-1)^{\epsilon_F \epsilon_G} G(F, H) .$$

The parity of the antibracket is

$$\epsilon[(F, G)] = \epsilon_F + \epsilon_G + 1 .$$

Now we have to compare these 3 different algebras, especially their Jacobi identities. For a comparision between the general formula (1) and the generalized Poisson bracket (7), the Jacobi identity must be converted to

$$\begin{aligned}
&(-1)^{km}[x_k,[x_l,x_m]] + (-1)^{lk}[x_l,[x_m,x_k]] + (-1)^{ml}[x_m,[x_k,x_l]] \\
&\quad = (-1)^{2km}(-1)^{kl}[[x_l,x_m],x_k] + (-1)^{2lk}(-1)^{lm}[[x_m,x_k],x_l] \\
&\qquad\qquad + (-1)^{2ml}(-1)^{mk}[[x_k,x_l],x_m] \\
&= [[x_l,x_m],x_k] + (-1)^{l(m+k)}[[x_m,x_k],x_l] + (-1)^{k(l+m)}[[x_k,x_l],x_m] = 0\,.
\end{aligned}$$

This is in agreement with (7) with the replacement $F = x_l$, $G = x_m$, $H = x_k$ and $\epsilon_F = l$, $\epsilon_G = m$, $\epsilon_H = k$.

In the case of the Batalin–Vilkovisky bracket the symmetry property reads as follows

$$(F,G) = -(-1)^{(\epsilon_F+1)(\epsilon_G+1)}(G,F)\,.$$

The parity of the bracket is

$$\epsilon[(F,G)] = \epsilon_F + \epsilon_G + 1\,.$$

So this means for the change in a “triple”

$$\begin{aligned}
(F,(G,H)) &= -(-1)^{(\epsilon_F+1)(\epsilon_{(G,H)}+1)}((G,H),F) \\
&= -(-1)^{(\epsilon_F+1)(\epsilon_G+\epsilon_H+2)}((G,H),F)\,.
\end{aligned}$$

The generalized Jacobi identity for a graded Lie algebra with a bracket of degree $\epsilon_{(\cdot\,,\cdot)} = 1$ is

$$\begin{aligned}
&(-1)^{(\epsilon_F+1)(\epsilon_H+1)}(F,(G,H)) + (-1)^{(\epsilon_G+1)(\epsilon_F+1)}(G,(H,F)) \\
&\qquad + (-1)^{(\epsilon_H+1)(\epsilon_G+1)}(H,(F,G)) = 0\,.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&-(-1)^{(\epsilon_F+1)(\epsilon_H+1)}(-1)^{(\epsilon_F+1)(\epsilon_G+\epsilon_H+2)}((G,H),F) \\
&-(-1)^{(\epsilon_G+1)(\epsilon_F+1)}(-1)^{(\epsilon_G+1)(\epsilon_H+\epsilon_F+2)}((H,F),G) \\
&-(-1)^{(\epsilon_H+1)(\epsilon_G+1)}(-1)^{(\epsilon_H+1)(\epsilon_F+\epsilon_G+2)}((F,G),H) = 0
\end{aligned}$$

$$\begin{aligned}
\Longleftrightarrow\ &((F,G),H) + (-1)^{(\epsilon_F+1)((\epsilon_G+1)+(\epsilon_H+1))}((G,H),F) \\
&\qquad + (-1)^{(\epsilon_H+1)((\epsilon_G+1)+(\epsilon_F+1))}((H,F),G) = 0\,.
\end{aligned} \tag{12}$$

So (12) is in agreement with (11).

In the case of the Gerstenhaber algebra one assumes a $\mathbb{Z}$-grading, so one cannot calculate modulo 2. The symmetry property and the parity of the Gerstenhaber bracket is

$$\begin{aligned}[v,w] &= -(-1)^{(\epsilon_v-1)(\epsilon_w-1)}[w,v] \\ \epsilon([v,w]) &= \epsilon_v + \epsilon_w - 1\,.\end{aligned}$$

The sign change in a "triple" is

$$[v,[w,y]] = -(-1)^{(\epsilon_v-1)(\epsilon_w+\epsilon_y-2)}[[w,y],v]\,.$$

The Jacobi identity which was given by Gerstenhaber [9] reads

$$\begin{aligned}(-1)^{(\epsilon_v-1)(\epsilon_y-1)}[[v,w],y] + (-1)^{(\epsilon_w-1)(\epsilon_v-1)}[[w,y],v]& \\ + (-1)^{(\epsilon_y-1)(\epsilon_w-1)}[[y,v],w] = 0\,.&\end{aligned} \tag{13}$$

This is on the one hand equivalent to

$$\begin{aligned}&[[v,w],y] + (-1)^{(\epsilon_v-1)((\epsilon_w-1)+(\epsilon_y-1))}[[w,y],v] \\ &\qquad +(-1)^{(\epsilon_y-1)((\epsilon_w-1)+(\epsilon_v-1))}[[y,v],w] = 0 \\ \Longleftrightarrow\ &[[v,w],y] + (-1)^{2(\epsilon_v-1)}(-1)^{(\epsilon_v-1)(\epsilon_w+\epsilon_y)}[[w,y],v] \\ &\qquad +(-1)^{2(\epsilon_y-1)}(-1)^{(\epsilon_y-1)(\epsilon_w+\epsilon_v)}[[y,v],w] = 0\end{aligned}$$

which is in agreement with (3). On the other hand (13) is equivalent to

$$\begin{aligned}&-(-1)^{(\epsilon_v-1)(\epsilon_y-1)}(-1)^{(\epsilon_y-1)(\epsilon_v+\epsilon_w-2)}[y,[v,w]] \\ &-(-1)^{(\epsilon_w-1)(\epsilon_v-1)}(-1)^{(\epsilon_v-1)(\epsilon_w+\epsilon_y-2)}[v,[w,y]] \\ &-(-1)^{(\epsilon_y-1)(\epsilon_w-1)}(-1)^{(\epsilon_w-1)(\epsilon_y+\epsilon_v-2)}[w,[y,v]] = 0 \\ \Longleftrightarrow\ &(-1)^{(\epsilon_y-1)(\epsilon_w-1)}[y,[v,w]] + (-1)^{(\epsilon_v-1)(\epsilon_y-1)}[v,[w,y]] \\ &\qquad + (-1)^{(\epsilon_w-1)(\epsilon_v-1)}[w,[y,v]] = 0 \\ \Longleftrightarrow\ &(-1)^{(\epsilon_y-1)(\epsilon_w+\epsilon_v)}[y,[v,w]] + [v,[w,y]] \\ &\qquad + (-1)^{(\epsilon_v-1)(\epsilon_w+\epsilon_y)}[w,[y,v]] = 0\end{aligned}$$

which is in agreement with the generalized Jacobi identity for a bracket with degree $\epsilon_{[\cdot,\cdot]} = -1$. Moreover it is in agreement with convention made in [17]. So despite their apparent differences the various forms of the Jacobi identity encountered in these different contexts are consistent.

An algebra can carry more than one grading. In field theory the $\mathbb{Z}_2$-grading is used to distinguish between fermionic and bosonic degrees of freedom, called

Grassmann parity; the $\mathbb{Z}$-grading is used to distinguish between fields and ghosts, called ghost number.

4. The Geometry of the Master Equation

The Alexandrov–Kontsevich–Schwarz–Zabronsky formalism [1] reflects the geometry of the master equation. The solution S of the classical master equation $(S, S) = 0$, which specifies a classical mechanical system, can be geometrically considered as a QP-manifold. This is a supermanifold N, equipped with an odd self-commuting vector field Q, $[Q, Q] = 0$ and an odd symplectic structure ω, which is Q-invariant. $\mathcal{F}(N)$ will denote the $\mathbb{Z}_2$-graded algebra of functions on this supermanifold.

A symplectic structure is defined as a closed, non-degenerate 2-form

$$\omega = \frac{1}{2}\, \mathrm{d}z^a \omega_{ab}(z)\, \mathrm{d}z^b$$

with the local coordinates

$$\{z^1, \ldots, z^n\} \text{ in } N, \quad \text{with parity} \quad \epsilon_a = \epsilon(z^a)\,.$$

The symplectic structure can be **even** or **odd** with respect to the $\mathbb{Z}_2$-grading of $\mathcal{F}(N)$.

In the even case the degree of the symplectic form is

$$\begin{aligned} \deg \omega_{ab} &= (\epsilon_a + \epsilon_b) \mod 2 \\ \omega_{ab} &= (-1)^{(\epsilon_a+1)(\epsilon_b+1)} \omega_{ba} \end{aligned}$$

while an odd symplectic structure satisfies

$$\begin{aligned} \deg \omega_{ab} &= (\epsilon_a + \epsilon_b + 1) \mod 2 \\ \omega_{ab} &= (-1)^{\epsilon_a \epsilon_b + 1} \omega_{ba}\,. \end{aligned}$$

The change of sign in $dz^a dz^b = -(-1)^{\epsilon_a \epsilon_b}\, \mathrm{d}z^b\, \mathrm{d}z^a$ is according to the Koszul sign rule. Define a bracket for functions $f, g \in \mathcal{F}(N)$

$$(f, g) = \frac{\partial_r f}{\partial z^a} \omega^{ab} \frac{\partial_l g}{\partial z^b}\,.$$

In the even case it corresponds to the generalized Poisson bracket, because its degree is 0. In the odd case the bracket equipps the algebra with a grading $\epsilon_{(\cdot,\cdot)} = 1$, like the BV-bracket. In general it will be called an **odd Poisson bracket**.

Associate to each $f \in \mathcal{F}(N)$ a vector field X_f

$$(f, g) = X_f(g)\,.$$

For an even bracket X_f has the same the same parity as f, while in the odd case X_f has the opposite parity to f. The odd case is of main interest, because it produces the BV-bracket.

The connection between the symplectic form, the vector field and the bracket is

$$\iota_{X_f}\omega = \mathrm{d}f$$
$$(f, g) =: X_f(g) = \iota_{X_f}\iota_{X_g}\omega\,.$$

The fundamental construction of a QP-manifold is given by the following definitions.

Definition 1. *The supermanifold N, which is considered in this context, can be constructed by associating to an ordinary manifold Σ its tangent bundle $T\Sigma$ and reversing the parity of the vector fields in the fiber. This is a simple turning of even (bosonic) vector fields into odd (fermionic) vector fields by a change of the corresponding variables, then $N = \Pi T\Sigma$. The construction is also possible with the cotangent bundle, which leads to $N = \Pi T^*\Sigma$.*

Definition 2. *A Q-manifold is a supermanifold N with an odd self-commuting vector field Q*

$$Q = Q^a \frac{\partial}{\partial z^a} \quad \textit{in local coordinates, with} \quad \deg Q^a = (\epsilon_a + 1) \mod 2$$
$$[Q, Q] = 0 \Longleftrightarrow Q^2 = 0$$

i. e. $[Q, Q] = 2Q^2 = 2(Q^b\partial_b Q^a)\partial_a$, so a Q-structure is the choice of a differential on $\mathcal{F}(N)$.

Definition 3. *A P-manifold is a supermanifold N with an odd symplectic structure ω*

$$\omega = \frac{1}{2}\,\mathrm{d}z^a \omega_{ab}(z)\,\mathrm{d}z^b, \qquad \deg \omega_{ab} = (\epsilon_a + \epsilon_b + 1) \mod 2\,.$$

Proposition 1. *There exists a Lie algebra homomorphism by the map*

$$f \to X_f$$
$$(f, g) \to [X_f, X_g]$$

which maps an odd Poisson or BV-bracket into a super commutator bracket.

Definition 4. *A vector field X can be represented in the form X_f iff ω is X-invariant, i. e.*

$$\mathcal{L}_X \omega = 0 \,.$$

Then X and ω are said to be compatible. Cartan'smagic formula for the Lie derivative is

$$\mathcal{L}_X = \mathrm{d}\iota_X + \iota_X \,\mathrm{d}$$

so

$$\mathcal{L}_X \omega = \mathrm{d}\iota_X \omega + \iota_X \,\mathrm{d}\omega$$

can be calculated to

$$\mathcal{L}_{X_f} \omega = \mathrm{d}(\iota_{X_f} \omega) = \mathrm{d}^2 f = 0$$

since $\mathrm{d}\omega = 0$ because ω is a symplectic form, and $\iota_{X_f} \omega = \mathrm{d}f$ for a Hamiltonian f and a Hamiltonian vector field X_f.

Definition 5. *A QP-manifold is a supermanifold with a compatible Q and P structures, i. e. the supermanifold is equipped with an odd self-commuting vectorfield Q and an odd symplectic structure ω which are compatible, $\mathcal{L}_Q \omega = 0$.*

The conclusion is that if

$$\iota_Q \omega = \mathrm{d}S$$

for some functions S, then Q is a Hamiltonian vector field and S is a Hamiltonian. This function S is even and fulfils $(S, S) = 0$, so every solution to the classical master equation determines a QP-structure and vice versa. The main point is that the geometrical structure, the QP-manifold, produces solutions of the master equation, which does not have to be solved in the usual way.

5. Quantization Procedures

Starting from a general gauge theory with well-defined algebraic structures, one has the choice between several procedures to quantize the theory. This is not always straightforward, each of these procedures has its own advantages and disadvantages.

The BV-algebra was quantized by Batalin and Vilkovisky [2], see also [3] for a more detailed description. This was done in the path integral formalism, with the help of several boundary conditions, to fulfil the postulates of a gauge theory. It leads to the quantum master equation, which ensures the BRST invariance of the partion function and works well for special cases. But for other models with a more complicated algebra, respectively open algebras like in string theory or the Sigma models, several obstructions occur.

Another method is deformation quantization, which takes the algebra structure and deforms it by expanding in a formal power series [4, 16]. The symmetry conditions lead to new bracket relations, involving star products, which make the quantization possible. Moreover, one wishes to consider more general manifolds than the simplectic ones, such as the Poisson manifolds. The geometry of these manifolds is more complicated and needs additional techniques. For a description see the books of Vaisman [20], and da Silva and Weinstein [17].

Cattaneo and Felder study the case of sigma models on manifolds with boundary; a special case of this construction yields the BV-action functional of the Poisson-sigma model on the disk [6]. Moreover they considered the Poisson Sigma in the AKSZ formalism [7].

QP-manifolds are related to strong homotopy Lie algebras, which are similar to Lie algebroids [17] and Gerstenhaber algebras [9], a modern generalization of Lie–Poisson algebras. The quantization of such a theory can be performed using deformation theory [18].

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