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COMPRESSED PRODUCT OF BALLS AND LOWER BOUNDARY ESTIMATES ON BERGMAN KERNELS

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Abstract. The image $B_{p^\sigma,q}$ of a product of balls $B_p \times B_q$ under a compression $c_\sigma(X,V) = (X, V(1 - {}^t\bar{X}X)^{\frac{\sigma}{2}})$ is called a compressed product of balls of exponent $\sigma \in \mathbb{R}$. The present note obtains the group $\mathrm{Aut}(B_{p^\sigma,q})$ of the holomorphic automorphisms and the $\mathrm{Aut}(B_{p^\sigma,q})$-orbit structure of $B_{p^\sigma,q}$ and its boundary $\partial B_{p^\sigma,q}$ for $\sigma > 1$. The Bergman completeness of $B_{p^\sigma,q}$ is verified by an explicit calculation of the Bergman kernel. As a consequence, local lower boundary estimates on the Bergman kernels of the bounded pseudoconvex domains are obtained, which are locally inscribed in $B_{p^\sigma,q}$ at a common boundary point.

For a strictly pseudoconvex domain $\mathcal{D} = \{z \in \mathbb{C}^n ;\, \rho(z) < 0\}$ with a smooth boundary, Fefferman [6] and Boutet de Monvel–Sjöstrand [2] have expanded the diagonal values $k_{\mathcal{D}}(z) := k_{\mathcal{D}}(z,z)$ of the Bergman kernel in the form $k_{\mathcal{D}}(z) = \varphi_{\mathcal{D}}(\rho)\rho^{-n-1} + \psi_{\mathcal{D}}(\rho)\log(-\rho)$, where $\varphi_{\mathcal{D}}(\rho)$ and $\psi_{\mathcal{D}}(\rho)$ are power series in the defining function $\rho = \rho(z)$.

Only few results are known for the boundary behavior of the Bergman kernel of a weakly pseudoconvex domain. For arbitrary $m = (m_1, \dots, m_n) \in \mathbb{N}^n$ let $\mathcal{E}_m := \{z \in \mathbb{C}^n ;\, \rho_m(z) = \sum_{j=1}^n |z_j|^{2m_j} - 1 < 0\}$, $z^o \in \partial\mathcal{E}_m$,

$$P_m := \{j \in \mathbb{N};\, z_j^o = 0 \text{ and } m_j > 1\}, \quad Q_m := \{j \in \mathbb{N};\, z_j^o \neq 0 \text{ or } m_j = 1\}.$$

Kamimoto has established in [11] the existence of an open subset $U \subset \mathbb{C}^n$ with $z^o \in \partial U$ and a real analytic function $\Phi_m \colon U \to \{r \in \mathbb{R};\, r > 0\}$, such that $k_{\mathcal{E}_m}(z) = \Phi_m(z)\rho_m(z)^{-\sum_{j\in P_m} m_j^{-1} - \mathrm{card}\, Q_m - 1}$ on U. If $P_m = \emptyset$ then $\Phi_m(z)$ is bounded around z^o, while $\lim_{z\to z^o}\Phi_m(z) = \infty$ for $P_m \neq \emptyset$.

In the spirit of Kamimoto's result, the present note provides local lower boundary estimates on the Bergman kernels of the bounded pseudoconvex domains, which are locally inscribed in a compressed product of balls. More precisely, on a product of balls

$$B_p \times B_q = \{X \in \mathrm{Mat}_{p,1}(\mathbb{C}), V \in \mathrm{Mat}_{q,1}(\mathbb{C});\ {}^t\bar{X}X < 1, {}^t\bar{V}V < 1\} \qquad (1)$$

let us consider a compression map

$$c_\sigma \colon B_p \times B_q \to B_p \times B_q, \quad c_\sigma(X,V) := \left(X, V(1 - {}^t\bar{X}X)^{\frac{\sigma}{2}}\right) \qquad (2)$$

of a positive real exponent σ. Its image

$$\begin{aligned} B_{p^\sigma,q} := c_\sigma(B_p \times B_q) = \{X \in B_p, Y \in B_q\,;\ &\rho_{p^\sigma,q}(X,Y) \\ &= {}^t\bar{Y}Y - (1 - {}^t\bar{X}X)^\sigma < 0\} \end{aligned} \qquad (3)$$

will be referred to as a **compressed product of balls**.

In Section 1 an explicit holomorphic action of the group $G = SU(p,1) \times U_q$ on $B_{p^\sigma,q}$ (cf. (5) from Lemma 1(ii)) is recognized. After describing the G-orbit decompositions of $B_{p^\sigma,q}$, $\partial B_{p^\sigma,q}$ and the geometry of the corresponding orbits, Proposition 1 establishes that the entire group $\mathrm{Aut}(B_{p^\sigma,q})$ of the holomorphic automorphisms of $B_{p^\sigma,q}$ is depleted by G.

The second section provides the diagonal values $k_{p^\sigma,q}(X,Y)$ of the Bergman kernel of $B_{p^\sigma,q}$. This is done by means of the $\mathrm{Aut}(B_{p^\sigma,q})$-transformation law of $k_{p^\sigma,q}$ and an explicit calculation of $k_{p^\sigma,q}$ at the canonical $\mathrm{Aut}(B_{p^\sigma,q})$-orbit representatives on $B_{p^\sigma,q}$ (cf. (8) from Lemma 1(iv)). If the diagonal values of the Bergman kernel of a bounded domain $\mathcal{D}$ tend to infinity, while approaching each of the boundary points, then $\mathcal{D}$ is called Bergman complete (cf. [1, 16]). Bremermann has shown in [4] that any Bergman complete domain is pseudoconvex and any bounded domain with a strictly plurisubharmonic exhaustion function is Bergman complete. However, there exist bounded pseudoconvex domains, which are not Bergman complete, e. g., the punctured disc $\Delta^* := \Delta \setminus \{0\}$, $\Delta := \{z \in \mathbb{C};\ |z| < 1\}$. This raises the problem of obtaining sufficient conditions for the Bergman completeness of bounded pseudoconvex domains. Blocki and Pflug [1] and, independently, Herbort [8] have shown that the existence of a plurisubharmonic exhaustion function of a bounded domain $\mathcal{D}$ is sufficient for its Bergman completeness. In [16] Zwonek proves that if a bounded pseudoconvex Reinhardt domain $\mathcal{D}$ does not admit an embedding $\varepsilon\colon \Delta^* \to \mathcal{D}$, which extends to $\varepsilon\colon \Delta \to \mathcal{D}$ with $\varepsilon(0) \in \partial\mathcal{D}$, then $\mathcal{D}$ is Bergman complete. The present note derives the Bergman completeness of the domain $B_{p^\sigma,q}$ from the explicit formula for $k_{p^\sigma,q}$.

A domain $\mathcal{D} \subset \mathbb{C}^n$ is locally geometrically approximated by a family $\mathcal{F}$ of domains in $\mathbb{C}^n$, if for any $Z \in \partial\mathcal{D}$ there exist a domain $\Omega_Z \in \mathcal{F}$ and a neighborhood $U(\Omega_Z)$ of Z on $\mathbb{C}^n$, such that $\mathcal{D} \cap U(\Omega_Z)$ is a domain in Ω_Z. A result of Diederich, Fornaess and Herbort (cf. [5] or Lemma 4) implies that the local geometric approximation by a family of bounded Bergman complete domains suffices for the Bergman completeness of a bounded pseudoconvex domain. The third section exposes the ultimate results of the present note in the form of local lower boundary estimates on the Bergman kernels of bounded pseudoconvex domains $\mathcal{D}$, which are locally inscribed in some $B_{p^\sigma,q}$ with $\sigma > 1$ at their common boundary point (cf. Corollary 2).

1. The Geometry and the Group of Holomorphic Automorphisms of a Compressed Product of Balls

Lemma 1. *Let $B_{p^\sigma,q}$ be a compressed product of balls of exponent $\sigma > 0$ (cf. (3)). Then:*

i) *$B_{p^\sigma,q}$ fibers by q-balls of variable radii $(1 - {}^t\bar{X}X)^{\frac{\sigma}{2}}$ over the p-ball B_p*

$$B_{p^\sigma,q} = \sqcup_{X \in B_p} B_q\left((1 - {}^t\bar{X}X)^{\frac{\sigma}{2}}\right);$$

ii) *the group*

$$G := SU(p,1) \times U_q \tag{4}$$

acts by holomorphic automorphisms on $B_{p^\sigma,q}$, according to the rule

$$g(X,Y) = ((AX+B)(CX+D)^{-1}, LY(CX+D)^{-\sigma}) \tag{5}$$

where

$$g = (N,L) \in G,\ N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(p,1), L \in U_q$$

and

$$(X,Y) \in B_{p^\sigma,q};$$

iii) *for any $(X_o, Y_o) \in B_{p^\sigma,q}$ the $(p+1)\times(p+1)$-matrix*

$$\xi^{X_o} := \begin{pmatrix} (I_p - X_o{}^t\bar{X}_o)^{-\frac{1}{2}} & -(I_p - X_o{}^t\bar{X}_o)^{-\frac{1}{2}}X_o \\ -(1 - {}^t\bar{X}_oX_o)^{-\frac{1}{2}}({}^t\bar{X}_o) & (1 - {}^t\bar{X}_oX_o)^{-\frac{1}{2}} \end{pmatrix} \tag{6}$$

belongs to $SU(p,1)$ and

$$\xi^{X_o}(X_o,Y_o) = \left(0, Y_o(1 - {}^t\bar{X}_oX_o)^{-\frac{\sigma}{2}}\right) \in \{0\} \times B_q; \tag{7}$$

iv) $B_{p^\sigma,q} = \sqcup_{s\in[0,1)} GM(s)$ *splits into a real one-parameter family of G-orbits with canonical representatives*

$$M(s) = (X = 0, Y = {}^t(s, 0, \dots, 0)); \tag{8}$$

v) *for* $s > 0$ *the orbits* $GM(s) = \sqcup_{X\in B_p} S^{2q-1}\left(s(1 - {}^t\bar{X}X)^{\frac{\sigma}{2}}\right)$ *fiber by* $(2q-1)$*-spheres of variable radii over the* p*-ball, while* $GM(0) = B_p \times \{0\}$ *is an ordinary* p*-ball.*

Proof: Claim (i) is immediate from the definition (3) of a compressed product of balls.

The indefinite special unitary group

$$SU(p,1) := \left\{ N = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SL_{p+1}(\mathbb{C});\ {}^t\bar{N}\begin{pmatrix} I_p & 0 \\ 0 & -1 \end{pmatrix} N = \begin{pmatrix} I_p & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

has defining equations

$$\begin{aligned} {}^t\bar{A}A - {}^t\bar{C}C &= I_p, & {}^t\bar{A}B - {}^t\bar{C}D &= 0, \\ {}^t\bar{B}B - {}^t\bar{D}D &= -1, & \det(N) &= 1 \end{aligned} \tag{9}$$

in terms of the matrices $A \in \mathrm{Mat}_{p,p}(\mathbb{C})$, $B \in \mathrm{Mat}_{p,1}(\mathbb{C})$, $C \in \mathrm{Mat}_{1,p}(\mathbb{C})$, $D \in \mathbb{C}$. The action of $N \in SU(p,1)$ on $X \in B_p$ is $N(X) := (AX + B)(CX + D)^{-1}$. Bearing in mind (9), one observes that $1 - {}^t\bar{N}(X)N(X) = (1 - {}^t\bar{X}X)|CX + D|^{-2}$ and verifies that g from (5) is a holomorphic bijection of $B_{p^\sigma,q}$. Straightforward verification of $N_2(N_1(X,Y)) = (N_2N_1)(X,Y)$ and $I_{p+1}(X,Y) = (X,Y)$ concludes the proof of (ii).

Concerning (iii), let us choose $T \in U_p$, rotating X_o to ${}^t\bar{T}X_o = {}^t(w_o, 0, \dots, 0)$ for some $w_o \in [0,1)$. Then

$$(I_p - X_o{}^t\bar{X}_o)^{-\frac{1}{2}} = T \begin{pmatrix} (1-w_o^2)^{-\frac{1}{2}} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix} {}^t\bar{T} \in \mathrm{Mat}_{p,p}(\mathbb{C})$$

is well defined positively definite Hermitian matrix. Expressing ξ^{X_o} from (6) by T and w_o, one verifies immediately the defining equations (9) of $SU(p,1)$.

For the proof of (iv) one multiplies the second component of (7) by an appropriate $L \in U_q$, in order to obtain $L\xi^{X_o}(X_o, Y_o) = (0, s_o)$ for some $s_o \in [0,1)$.

Towards the description of the G-orbits on $B_{p^\sigma,q}$, we note that the projection $\Pi\colon B_{p^\sigma,q} \to B_p$, $\Pi(X,Y) = X$ commutes with the $SU(p,1)$-actions on $B_{p^\sigma,q}$ and B_p. Thus, for an arbitrary $X \in B_p$ there exists $N \in SU(p,1)$, such that $X = N(0) = N\Pi M(s) = \Pi NM(s) \in \Pi GM(s)$ for all $s \in [0,1)$. In other words, any $\Pi_s\colon GM(s) \to B_p$ is surjective and the fibers

$\Pi_s^{-1}(X) = \Pi_s^{-1}(N(0)) = \Pi_s^{-1}(BD^{-1}) = \{(BD^{-1}, L^t(s,0,\dots,0)D^{-\sigma});\ L \in U_q\} \simeq \{Y \in \mathrm{Mat}_{q,1}(\mathbb{C});\ {}^t\bar{Y}Y = (s|D|^{-\sigma})^2\}$ are $(2q-1)$-spheres, which degenerate to the origin $0 \in \mathbb{C}^q$ for $s = 0$. □

The next lemma describes some geometric and analytic properties of the boundary $\partial B_{p^\sigma,q}$. The case of an ordinary ball $B_{p^1,q} \equiv B_{p+q}$ is well known.

Lemma 2. *Let* $\partial B_{p^\sigma,q} = \{X \in \bar{B}_p, Y \in \bar{B}_q;\ \rho_{p^\sigma,q}(X,Y) = {}^t\bar{Y}Y - (1 - {}^t\bar{X}X)^\sigma = 0\}$ *be the boundary of a compressed product of balls* $B_{p^\sigma,q}$*. Then:*

i) *in the notations of part (ii) of Lemma* 1*, any* $g \in G$ *extends to a real analytic diffeomorphism of* $\partial B_{p^\sigma,q}$*;*

ii) *for* $\sigma > 1$ *the boundary* $\partial B_{p^\sigma,q}$ *is of class* C^1 *and the singular boundary locus is the orbit*

$$\partial B_{p^\sigma,q} \cap \{Y = 0\} = S^{2p-1} \times \{0\} = U_p(N)$$

of

$$N = (X = {}^t(1,0,\dots,0), Y = 0) \in \partial B_{p^\sigma,q};$$

iii) *the smooth boundary locus*

$$(\partial B_{p^\sigma,q})^o = \sqcup_{X \in B_p} S^{2q-1}\left((1 - {}^t\bar{X}X)^{\frac{\sigma}{2}}\right) \tag{10}$$

fibers by $(2q-1)$*-spheres of variable radii over the* p*-ball and constitutes the orbit*

$$(\partial B_{p^\sigma,q})^o = (SU(p,1) \times U_q)M \tag{11}$$

of

$$M = (X = 0, Y = {}^t(1,0,\dots,0)) \in (\partial B_{p^\sigma,q})^o .$$

Proof: We claim that ξ^{X_o} from (6), Lemma 1 (iii) extends in a neighborhood of $\partial B_{p^\sigma,q}$. Otherwise, there is $X_1 \in \mathrm{Mat}_{p,1}(\mathbb{C})$ with ${}^t\bar{X}_oX_1 = 1$ and ${}^t\bar{X}_1X_1 = 1$. Cauchy–Schwarz inequality for the standard Hermitian inner product in $\mathrm{Mat}_{p,1}(\mathbb{C})$ implies that $1 = {}^t\bar{X}_oX_1 \leq \sqrt{{}^t\bar{X}_oX_o}\sqrt{{}^t\bar{X}_1X_1} = \sqrt{{}^t\bar{X}_oX_o} < 1$, which is an absurd. This justifies the extendability of ξ^{X_o} over $\partial B_{p^\sigma,q}$. For an arbitrary $g = (N, L)$ with $N \in SU(p,1)$, let $X_o := N^{-1}(0)$ where $0 \in \mathbb{C}^p$ stands for the origin of B_p. Then $\xi^{X_o}N^{-1} \in \mathrm{Aut}_{\check{o}}(B_p) = U_p$ acts linearly on B_p and extends to $\mathrm{Mat}_{p,1}(\mathbb{C}) \supset \bar{B}_p$. Similarly, $L \in U_q$ induces a diffeomorphism of $\partial B_{p^\sigma,q}$.

Towards the verification of (ii), one calculates that the real gradient

$$\begin{aligned}&\left(\frac{\partial\rho_{p^\sigma,q}}{\partial(\mathrm{r}x_1)},\frac{\partial\rho_{p^\sigma,q}}{\partial(\mathrm{i}x_1)},\ldots,\frac{\partial\rho_{p^\sigma,q}}{\partial(\mathrm{r}x_p)},\frac{\partial\rho_{p^\sigma,q}}{\partial(\mathrm{i}x_p)},\ldots,\right.\\&\left.\frac{\partial\rho_{p^\sigma,q}}{\partial(\mathrm{r}y_1)},\frac{\partial\rho_{p^\sigma,q}}{\partial(\mathrm{i}y_1)},\ldots,\frac{\partial\rho_{p^\sigma,q}}{\partial(\mathrm{r}y_q)},\frac{\partial\rho_{p^\sigma,q}}{\partial(\mathrm{i}y_q)}\right)\\&=\left(2\sigma(\mathrm{r}x_1)(1-{}^t\bar{X}X)^{\sigma-1},2\sigma(\mathrm{i}x_1)(1-{}^t\bar{X}X)^{\sigma-1},\ldots,2\mathrm{r}y_q,2\mathrm{i}y_q\right)\end{aligned}$$

vanishes exactly when $X\in S^{2p-1}=U_p(N)$ and $Y=0\in\mathbb{C}^q$.
As an immediate consequence of (ii) and the defining equation (10) of $\partial B_{p^\sigma,q}$, follows. The argument of Lemma 1 (ii) reveals the G-invariance of $(B_{p^\sigma,q})^o$, whereas $GM\subseteq(B_{p^\sigma,q})^o$. Conversely, for any $(X_o,Y_o)\in(B_{p^\sigma,q})^o$ the proof of Lemma 1 (iii) provides $\xi^{X_o}\in SU(p,1)$ with $\xi^{X_o}(X_o,Y_o)=(0,Y_1)$, $Y_1=Y_o(1-{}^t\bar{X}_oX_o)^{-\frac{\sigma}{2}}$. Then an appropriate $L\in U_q$ brings $Y_1\in S^{2q-1}$ to ${}^t(1,0,\ldots,0)$. □

We are now in the position to describe the reduction of the identity component $\mathrm{Aut}(B_p\times B_q)_1=SU(p,1)\times SU(q,1)$, caused by a compression map c_σ (cf. (2)).

Proposition 1.

i) *The holomorphic automorphisms, fixing the origin* $\check{o}=(X=0,Y=0)$ *of a compressed product of balls* $B_{p^\sigma,q}$ *of exponent* $\sigma>1$ *form the group*

$$\mathrm{Aut}_{\check{o}}(B_{p^\sigma,q})=\mathrm{Aut}_{\check{o}}(B_{p^\sigma,q})\cap G=S(U_p\times U_1)\times U_q\simeq U_p\times U_q\,. \tag{12}$$

ii) *The group of the holomorphic automorphisms of a compressed product of balls* $B_{p^\sigma,q}$ *of exponent* $\sigma>1$ *is*

$$\mathrm{Aut}(B_{p^\sigma,q})=SU(p,1)\times U_q. \tag{13}$$

Proof:

i) By definition (3), the domain $B_{p^\sigma,q}$ is circular, i. e., invariant under the action $(X,Y)\mapsto(\mathrm{e}^{\mathrm{i}\alpha}X,\mathrm{e}^{\mathrm{i}\alpha}Y)$ of $U_1=\{\mathrm{e}^{\mathrm{i}\alpha};\,\alpha\in[0,2\pi)\}$. The holomorphic automorphisms, fixing the origin of a bounded circular domain act by unitary linear transformations on the ambient Euclidean space (cf. [10, 13]). Therefore, any $\varphi\in\mathrm{Aut}_{\check{o}}(B_{p^\sigma,q})\subseteq U_{p+q}$ extends to a real analytic diffeomorphism of the boundary $\partial B_{p^\sigma,q}$, preserving the smooth and the singular boundary loci. More precisely, if $\varphi=\begin{pmatrix}A&B\\C&D\end{pmatrix}\in U_{p+q}$ with $A\in\mathrm{Mat}_{p,p}(\mathbb{C})$, $B\in\mathrm{Mat}_{p,q}(\mathbb{C})$, $C\in\mathrm{Mat}_{q,p}(\mathbb{C})$, $D\in\mathrm{Mat}_{q,q}(\mathbb{C})$ transforms any $(X,Y=0)\in S^{2p-1}\times\{0\}$ in $(AX,CX=0)\in S^{2p-1}\times\{0\}$, then $C=0$. Further, ${}^t\bar{\varphi}\varphi=I_{p+q}$ implies that $B=0$, $A\in U_p$

and $D \in U_q$, whereas $\mathrm{Aut}_{\check{o}}(B_{p^\sigma,q}) \subseteq U_p \times U_q$. The obvious inclusion $U_p \times U_q = \mathrm{Aut}_{\check{o}}(B_{p^\sigma,q}) \cap G \subseteq \mathrm{Aut}_{\check{o}}(B_{p^\sigma,q})$ concludes the proof of (i).

ii) Let us suppose that $\varphi \in \mathrm{Aut}(B_{p^\sigma,q})$ maps $\check{o} = M(0)$ in $(X_o, Y_o) \in B_{p^\sigma,q}$. According to Lemma 1 (iv), there exist $g \in G$ and $s \in [0,1)$, such that $g(X_o, Y_o) = M(s)$ (cf. (8)). If $s = 0$ then $g\varphi \in \mathrm{Aut}_{\check{o}}(B_{p^\sigma,q}) \subset G$, so that $\varphi \in G$ and $\mathrm{Aut}(B_{p^\sigma,q}) = G$.
The opposite assumption $s \neq 0$ implies that $GM(s) \subseteq \mathrm{Aut}(B_{p^\sigma,q})M(0)$, whereas $2p+2q-1 = \dim_{\mathbb{R}} GM(s) \leq \dim_{\mathbb{R}} \mathrm{Aut}(B_{p^\sigma,q})M(0)$. According to Braun, Kaup and Upmeier's article [3], the $\mathrm{Aut}(B_{p^\sigma,q})$-orbit of the origin $M(0)$ of the bounded circular domain $B_{p^\sigma,q}$ is a bounded symmetric domain, embedded as a closed complex analytic submanifold of $B_{p^\sigma,q}$. Therefore, $\mathrm{Aut}(B_{p^\sigma,q})M(0) = B_{p^\sigma,q}$ is a Hermitian symmetric space of noncompact type. In particular, the boundary $\partial B_{p^\sigma,q}$ is smooth. This contradicts Lemma 2 (ii), thus justifying that $s = 0$.

□

2. The Bergman Kernel of a Compressed Product of Balls

Lemma 3. *At the canonical* $\mathrm{Aut}(B_{p^\sigma,q})$*-orbit representatives* $M(s)$ *of* $B_{p^\sigma,q}$ *with* $\sigma > 0$ *(cf.* (8) *from Lemma* 1*), the diagonal values of the Bergman kernel are*

$$k_{p^\sigma,q}(M(s)) = C_o \left[\sum_{j=0}^{p} F_j(\sigma) s^{2j} (1-s^2)^{p-j} \right] \times \left[(1-s^2)^{q+1} - s^{2(q+1)} \right] (1-s^2)^{-p-q-1} (1-2s^2)^{-1} \tag{14}$$

where

$$C_o := (-1)^{[\frac{p+1}{2}]+[\frac{q+1}{2}]} \pi^{-p-q}, \qquad F_p(\sigma) := (p+q)!\sigma^p \tag{15}$$

$$F_j(\sigma) := \left[\prod_{l=1}^{q} (j+l) \right] \left\{ \sum_{i=j}^{p-1} \left[\prod_{l=1}^{p-i} (\sigma q + i + l) \right] a_{i,j}(\sigma) + a_{p,j}(\sigma) \right\} \sigma^j, \quad \textit{for all } \; 0 \leq j \leq p-1\,, \tag{16}$$

are expressed by the recursively defined $a_{i,j}$,

$$a_{i,0}(\sigma) := 0, \quad \textit{for all} \;\; i \in \mathbb{N}$$
$$a_{i,1}(\sigma) := \prod_{l=1}^{i-1} (\sigma - l), \quad \textit{for all} \;\; i \in \mathbb{N} \setminus \{1\} \tag{17}$$
$$a_{i,i}(\sigma) := i!, \quad \textit{for all} \;\; i \in \mathbb{N} \cup \{0\}$$

$$
\begin{aligned}
a_{i,j}(\sigma) &:= (\sigma j - i + 1)a_{i-1,j}(\sigma) + j a_{i-1,j-1}(\sigma), \\
&\textit{for all} \;\; i \geq 3, 2 \leq j \leq i-1 \,.
\end{aligned}
\tag{18}
$$

Proof: The compressed product of balls $B_{p^\sigma,q}$ is a bounded Reinhardt domain, i. e., $B_{p^\sigma,q}$ is invariant under the action ${}^t(x_1,\dots,x_p) \times {}^t(y_1,\dots,y_q) \mapsto {}^t(\mathrm{e}^{\mathrm{i}\alpha_1}x_1,\dots,\mathrm{e}^{\mathrm{i}\alpha_p}x_p) \times {}^t(\mathrm{e}^{\mathrm{i}\beta_1}y_1,\dots,\mathrm{e}^{\mathrm{i}\beta_q}y_q)$ of the torus $T^{p+q} := \underbrace{U_1 \times \cdots \times U_1}_{p+q}$, $\mathrm{e}^{\mathrm{i}\alpha_j}, \mathrm{e}^{\mathrm{i}\beta_k} \in U_1$. Therefore the monomials $x_1^{i_1}\dots x_p^{i_p} y_1^{j_1}\dots y_q^{j_q}$ with $i_l, j_l \in \mathbb{N} \cup \{0\}$ form an orthogonal basis of the Hilbert space of the holomorphic square integrable functions $\mathcal{O}_{B_{p^\sigma,q}} \cap L^2_{B_{p^\sigma,q}}$ and

$$
k_{p^\sigma,q}(M(s)) = \sum_{j=0}^{\infty} \frac{s^{2j}}{\|y_1^j\|^2_{B_{p^\sigma,q}}}\,.
$$

Here $\|y_1^j\|^2_{B_{p^\sigma,q}} := \left(\frac{\mathrm{i}}{2}\right) \int_{B_{p^\sigma,q}} |y_1|^{2j}\, \mathrm{d}x_1 \wedge \mathrm{d}\bar{x}_1 \wedge \cdots \wedge \mathrm{d}x_p \wedge \mathrm{d}\bar{x}_p \wedge \mathrm{d}y_1 \wedge \mathrm{d}\bar{y}_1 \wedge \cdots \wedge \mathrm{d}y_q \wedge \mathrm{d}\bar{y}_q$. The real analytic diffeomorphism $B_{p^\sigma,q} \ni (X,Y) \mapsto \left(X, Z := Y(1 - {}^t\bar{X}X)^{\frac{1-\sigma}{2}}\right) \in B_{p+q}$, followed by polar coordinate changes $x_l := u_l e^{\mathrm{i}\theta_l}$, $z_m := v_m e^{\mathrm{i}\tau_m}$ where $\theta_l, \tau_m \in [0, 2\pi)$, $u_l, v_m \in \mathbb{R}$, $r := \sum_{l=1}^p u_l^2 + \sum_{m=1}^q v_m^2 < 1$, yields

$$
\begin{aligned}
&\|y_1^j\|^2_{B_{p^\sigma,q}} \\
&= C_o^{-1} \int\limits_{r<1} \left(1 - \sum_{l=1}^p u_l^2\right)^{(\sigma-1)(j+q)} v_1^{2j}\, \mathrm{d}v_1^2 \wedge \mathrm{d}v_2^2 \wedge \cdots \wedge \mathrm{d}v_q^2 \wedge \mathrm{d}u_1^2 \wedge \cdots \wedge \mathrm{d}u_p^2 \\
&= C_o^{-1} \prod_{m=1}^q (j+m)^{-1} \int\limits_{\sum_{l=1}^p u_l^2 < 1} \left(1 - \sum_{l=1}^p u_l^2\right)^{\sigma(j+q)} \mathrm{d}u_1^2 \wedge \cdots \wedge \mathrm{d}u_p^2 \\
&= C_o^{-1} \prod_{m=1}^q (j+m)^{-1} \left\{ \prod_{l=1}^p [\sigma(j+q) + l]^{-1} \right\}
\end{aligned}
$$

bearing in mind the fact that $\displaystyle\int\limits_{\sum_{l=k}^m w_l^2 < \varepsilon} \left(\varepsilon - \sum_{l=k}^m w_l^2\right)^{\delta(n+k-1)} \mathrm{d}w_k^2 \wedge \cdots \wedge \mathrm{d}w_m^2 =$

$$
[\delta(n+k-1)+1]^{-1} \int\limits_{\sum_{l=k+1}^m w_l^2 < \varepsilon} \left(\varepsilon - \sum_{l=k+1}^m w_l^2\right)^{\delta(n+k-1)+1} \mathrm{d}w_{k+1}^2 \wedge \cdots \wedge \mathrm{d}w_m^2
$$

for arbitrary $k, m, n \in \mathbb{N}$, $k < m$, real positive ε, δ and real variables

$w_k, w_{k+1}, \ldots, w_m$. Then

$$\begin{aligned} k_{p^\sigma,q}(M(s)) &= C_o \sum_{j=0}^{\infty} \left\{ \prod_{m=1}^{q} (j+m) \prod_{l=1}^{p} [\sigma(j+q)+l] \right\} s^{2j} \\ &= C_o \frac{\mathrm{d}^q}{\mathrm{d}\theta^q} \left[\frac{\mathrm{d}^p}{\mathrm{d}t^p} \left(\frac{t^{\sigma q+p}}{1-t^\sigma} \right) \Big|_{t=\sqrt[\sigma]{\theta}} \right] \Big|_{\theta=s^2} \end{aligned}$$

according to $\dfrac{t^{\sigma q+p}}{1-t^\sigma} = \sum_{j=0}^{\infty} t^{\sigma(j+q)+p}$. By an induction on i one verifies that

$$\frac{\mathrm{d}^i}{\mathrm{d}t^i}(1-t^\sigma)^{-1} = \sum_{j=0}^{i} a_{i,j}(\sigma)\sigma^j t^{\sigma j-i}(1-t^\sigma)^{-j-1} \qquad \text{for all } \; i \in \mathbb{N} \cup \{0\}$$

where $a_{i,j}(\sigma)$ are defined by (17) and (18). Consequently,

$$\begin{aligned} &\frac{\mathrm{d}^p}{\mathrm{d}t^p} \left[(1-t^\sigma)^{-1} t^{\sigma q+p} \right] \Big|_{t=\sqrt[\sigma]{\theta}} \\ &= \sum_{j=0}^{p-1} \left\{ \sum_{i=j}^{p-1} \left[\prod_{l=1}^{p-i} (\sigma q+i+l) \right] a_{i,j}(\sigma) + a_{p,j}(\sigma) \right\} \\ &\qquad \times \sigma^j \theta^{j+q}(1-\theta)^{-j-1} + p!\sigma^p \theta^{p+q}(1-\theta)^{-p-1}. \end{aligned}$$

Making use of

$$\frac{\mathrm{d}^q}{\mathrm{d}\theta^q} \left[\theta^{j+q}(1-\theta)^{-j-1} \right] \Big|_{\theta=s^2} = \left[\prod_{l=1}^{q} (j+l) \right] s^{2j}(1-s^2)^{-j-1} \left[\sum_{m=0}^{q} \left(\frac{s^2}{1-s^2} \right)^m \right]$$

(15) and (16), one obtains (14). □

Proposition 2. *At an arbitrary point* (X,Y) *of* $B_{p^\sigma,q}$ *with* $\sigma > 1$, *the diagonal values of the Bergman kernel*

$$\begin{aligned} k_{p^\sigma,q}(X,Y) &= C_o \left\{ \sum_{j=0}^{p} F_j(\sigma) \left[\frac{{}^t\bar{Y}Y}{(1-{}^t\bar{X}X)^\sigma} \right]^j \left[1 - \frac{{}^t\bar{Y}Y}{(1-{}^t\bar{X}X)^\sigma} \right]^{p-j} \right\} \\ &\times \left\{ \left[1 - \frac{{}^t\bar{Y}Y}{(1-{}^t\bar{X}X)^\sigma} \right]^{q+1} - \left[\frac{{}^t\bar{Y}Y}{(1-{}^t\bar{X}X)^\sigma} \right]^{q+1} \right\} \\ &\times \left[1 - \frac{{}^t\bar{Y}Y}{(1-{}^t\bar{X}X)^\sigma} \right]^{-p-q-1} \left[1 - 2\frac{{}^t\bar{Y}Y}{(1-{}^t\bar{X}X)^\sigma} \right]^{-1} (1-{}^t\bar{X}X)^{-q\sigma-p-1} \end{aligned}$$

in the notations from (15), (16), (17), (18), *Lemma* 3.

Proof: The $\mathrm{Aut}(B_{p^\sigma,q})$-transformation law of the Bergman kernel states that

$$k_{p^\sigma,q}(Z_o) = k_{p^\sigma,q}(g(Z_o))|\det J^{\mathbb{C}}g|_{Z_o}|^2$$

for arbitrary $Z_o = (X_o, Y_o) \in B_{p^\sigma,q}$ and $g \in \mathrm{Aut}(B_{p^\sigma,q})$ with holomorphic Jacobian matrix $J^{\mathbb{C}}g$ (cf. [12]). According to Lemma 1(iii) and the proof of Lemma 1(iv), there exist $\xi^{X_o} \in SU(p,1)$, given by (6) and $L \in U_q$, such that$(\xi^{X_o}, L)(X_o, Y_o) = M(s_o)$ for $s_o = \left[{}^t\bar{Y}_oY_o\left(1 - {}^t\bar{X}_oX_o\right)^{-\sigma}\right]^{\frac{1}{2}} \in [0,1)$. It is straightforward that $g^{Z_o} := (\xi^{X_o}, L)$ has

$$J^{\mathbb{C}}g^{Z_o}|_{Z_o} = \begin{pmatrix} (I_p - X_o{}^t\bar{X}_o)^{-\frac{1}{2}}(1 - {}^t\bar{X}_oX_o)^{-\frac{1}{2}} & 0_{p,q} \\ * & L(1 - {}^t\bar{X}_oX_o)^{-\frac{\sigma}{2}} \end{pmatrix} \in \mathrm{Mat}_{p+q,p+q}(\mathbb{C})\,.$$

Plugging in $X_o = T^t(w_o, 0, \ldots, 0)$ for appropriate $T \in U_p$, $w_o \in [0,1)$, one observes that $\det(I_p - X_o{}^t\bar{X}_o) = 1 - w_o^2 = 1 - {}^t\bar{X}_oX_o$. Consequently, $|\det J^{\mathbb{C}}g^{Z_o}_{Z_o}|^2 = (1 - {}^t\bar{X}_oX_o)^{-q\sigma-p-1}$. Putting together with Lemma 3, one obtains the announced $k_{p^\sigma,q}(X_o, Y_o)$. $\square$

A bounded domain $\mathcal{D} \subset \mathbb{C}^n$ is Bergman complete if for any $z^o \in \partial\mathcal{D}$ and any $z_n \in \mathcal{D}$ with $\lim_{n\to\infty} z_n = z^o$, the diagonal values of the Bergman kernel $\lim_{z_n\to z^o} k_{\mathcal{D}}(z_n) = \infty$.

Corollary 1. *Let $B_{p^\sigma,q}$ be a compressed product of balls of exponent $\sigma > 1$. Then:*

i) *for any smooth $(X_o, Y_o) \in \partial B_{p^\sigma,q}$ there exist a neighborhood U of (X_o, Y_o), a positive real constant c and a real analytic function $\Phi_{p^\sigma,q} : U \to (0,c)$, such that*

$$k_{p^\sigma,q}(X,Y) = \Phi_{p^\sigma,q}(X,Y)\left[1 - \frac{{}^t\bar{Y}Y}{(1 - {}^t\bar{X}X)^\sigma}\right]^{-p-q-1} \tag{19}$$

for all $(X,Y) \in B_{p^\sigma,q} \cap U$;

ii) *for any singular $(X_1, 0) \in \partial B_{p^\sigma,q}$ there exist a neighborhood U of $(X_1, 0)$, a positive real constant c and a real analytic function $\Psi_{p^\sigma,q} : U \to (0,c)$, such that*

$$k_{p^\sigma,q}(X,Y) = \Psi_{p^\sigma,q}(X,Y)\left[1 - \frac{{}^t\bar{Y}Y}{(1 - {}^t\bar{X}X)^\sigma}\right]^{-p-q-1}(1 - {}^t\bar{X}X)^{-\sigma q-p-1} \tag{20}$$

for all $(X,Y) \in B_{p^\sigma,q} \cap U$. In particular, $B_{p^\sigma,q}$ is Bergman complete.

Proof: Let $B_{p^\sigma,q} = \left\{X \in B_p, Y \in B_q;\, \tau(X,Y) = \frac{{}^t\bar{Y}Y}{(1-{}^t\bar{X}X)^\sigma} < 1\right\}$, so that

$$k_{p^\sigma,q} = \frac{C_o\left[\sum_{j=0}^{p} F_j(\sigma)\tau^j(1-\tau)^{p-j}\right]\left[\sum_{j=0}^{q}\tau^j(1-\tau)^{q-j}\right]}{(1-\tau)^{p+q+1}\left(1-{}^t\bar{X}X\right)^{\sigma q+p+1}}.$$

Then $\Psi_{p^\sigma,q} := k_{p^\sigma,q}\,(1-\tau)^{p+q+1}\left(1-{}^t\bar{X}X\right)^{\sigma q+p+1}$ is a bounded real analytic positively valued function on the closure $\bar{B}_{p^\sigma,q}$. In particular, if (X_o, Y_o) is a smooth boundary point of $B_{p^\sigma,q}$ then ${}^t\bar{X}_oX_o < 1$ (cf. Lemma 2(iii)), so that the function $\Phi_{p^\sigma,q} := \left(1-{}^t\bar{X}X\right)^{-\sigma q-p-1}\Psi_{p^\sigma,q}$ satisfies Corollary 1 (i). □

3. Lower Boundary Estimates on Bergman Kernels of Bounded Pseudoconvex Domains

A domain $\mathcal{D} \subset \mathbb{C}^n$ is locally geometrically approximated by a family $\mathcal{F}$ of domains in $\mathbb{C}^n$ if for any $z \in \partial\mathcal{D}$ there exist a domain $\Omega_z \in \mathcal{F}$ and a neighborhood $U(\Omega_z)$ of Z such that $\mathcal{D} \cap \Omega_z$ is a domain in Ω_z. We need a result of Diederich, Fornaess and Herbort (cf. [5]) in the following slightly modified form:

Lemma 4. (Diederich, Fornaess, and Herbort [5]) *Let $\mathcal{D} \subset \mathbb{C}^n$ be a bounded pseudoconvex domain and U be a neighborhood of $Z_o \in \partial\mathcal{D}$, such that $\mathcal{D} \cap U$ is a subdomain in a bounded domain $\mathcal{D}_o \subset \mathbb{C}^n$. Then for any neighborhood $V \subset\subset U$ of Z_o there exists a constant $c \in (0,1]$ such that*

$$k_{\mathcal{D}}(z) \geq c k_{\mathcal{D}_o}(z)$$

for all $z \in \mathcal{D} \cap V$. In particular, if a bounded pseudoconvex domain $\mathcal{D}$ admits a local geometric approximation by a family $\mathcal{F}$ of bounded Bergman complete domains, then $\mathcal{D}$ is Bergman complete.

Proof: The aforementioned theorem from [5] asserts that for any $V \subset\subset U$, and $Z_o \in V$, there exists a constant $c \in (0,1]$, such that $k_{\mathcal{D}}(z) \geq c k_{\mathcal{D}\cap U}(z)$ for arbitrary $z \in \mathcal{D} \cap V$. As far as $\mathcal{D} \cap U$ is a domain in $\mathcal{D}_o$, a result of Kobayashi from [9] applies to provide $k_{\mathcal{D}\cap U}(z) \geq k_{\mathcal{D}_o}(z)$ for $z \in \mathcal{D} \cap U$. □

Putting together Corollary 1 and Lemma 4, one obtains the following

Corollary 2. *Let $\mathcal{D} \subset \mathrm{Mat}_{p,1}(\mathbb{C}) \times \mathrm{Mat}_{q,1}(\mathbb{C}) \simeq \mathbb{C}^{p+q}$ be a bounded pseudoconvex domain, $B_{p^\sigma,q}$ be a compressed product of balls of exponent $\sigma > 1$ and $Z_o = (X_o, Y_o) \in \partial\mathcal{D} \cap \partial B_{p^\sigma,q}$ be a common boundary point. Suppose that there is a neighborhood U of Z_o with connected $\mathcal{D} \cap U \subset B_{p^\sigma,q}$.*

(i) If Z_o is a smooth boundary point of $B_{p^\sigma,q}$ then there exists a neighborhood $V \subset\subset U$ and a real analytic function $\Phi_{p^\sigma,q}\colon V \to (0,\infty)$, such that

$$k_{\mathcal{D}}(Z) = \Phi_{p^\sigma,q}(X,Y)\left[1 - \frac{{}^t\bar{Y}Y}{(1 - {}^t\bar{X}X)^\sigma}\right]^{-p-q-1} \tag{21}$$

for all $(X,Y) \in \mathcal{D} \cap V$.
(ii) If Z_o is a singular boundary point of $B_{p^\sigma,q}$ then there exist $V \subset\subset U$ and a real analytic function $\Psi_{p^\sigma,q}\colon V \to (0,\infty)$, such that

$$k_{\mathcal{D}}(Z) = \Psi_{p^\sigma,q}(X,Y)\left[1 - \frac{{}^t\bar{Y}Y}{(1 - {}^t\bar{X}X)^\sigma}\right]^{-p-q-1} (1 - {}^t\bar{X}X)^{-\sigma q-p-1} \tag{22}$$

for all $(X,Y) \in \mathcal{D} \cap V$.
In particular, if $\mathcal{D}$ admits a local geometric approximation by the family of the compressed products of balls $B_{p^\sigma,q}$ of exponent $\sigma > 1$, then $\mathcal{D}$ is Bergman complete.

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