

THE STABILITY PROBLEM AND THE EXISTENCE OF PERIODIC ORBITS IN THE HEAVY TOP DYNAMICS

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Abstract. The stability problem and the existence of periodic orbits for the heavy top dynamics are discussed and some of its properties are pointed out.

1. Introduction

In the last time there was a great deal of interest in the study of the heavy top dynamics. The goal of our paper is to present some old and new aspects from its stability problem and from the problem of the existence of its periodic orbits.

2. The Heavy Top

The heavy top is by definition a rigid body which moves around a fixed point in the 3-dimensional space. The rigidity means that the distances between the points when the body moves remain the same, so they are fixed.

The dynamics of the heavy top is described by the following set of differential equations, usually called **Euler equations**:

$$\begin{aligned} \dot{m}_1 &= a_1 m_2 m_3 + mgl(\gamma_2 \chi_3 - \gamma_3 \chi_2) \\ \dot{m}_2 &= a_2 m_1 m_3 + mgl(\gamma_3 \chi_1 - \gamma_1 \chi_3) \\ \dot{m}_3 &= a_3 m_1 m_2 + mgl(\gamma_1 \chi_2 - \gamma_2 \chi_1) \\ \dot{\gamma}_1 &= \frac{m_3 \gamma_2}{I_3} - \frac{m_2 \gamma_3}{I_2} \\ \dot{\gamma}_2 &= \frac{m_1 \gamma_3}{I_1} - \frac{m_3 \gamma_1}{I_3} \end{aligned} \tag{1}$$

$$\dot{\gamma}_3 = \frac{m_2\gamma_1}{I_2} - \frac{m_1\gamma_2}{I_1}$$

where

$$a_1 = \frac{1}{I_3} - \frac{1}{I_2}, \quad a_2 = \frac{1}{I_1} - \frac{1}{I_3}, \quad a_3 = \frac{1}{I_2} - \frac{1}{I_1}$$

and we suppose as usually that:

$$I_1 \geq I_2 \geq I_3 > 0.$$

In all that follows we shall be interested in the following two particular cases:

i) The center of mass lies on the axis of symmetry in the body, i. e.,

$$\chi = (0, 0, 1).$$

In this case the equations of motion takes the following form:

$$\begin{aligned} \dot{m}_1 &= a_1 m_2 m_3 + mgl\gamma_2 \\ \dot{m}_2 &= a_2 m_1 m_3 - mgl\gamma_1 \\ \dot{m}_3 &= a_3 m_1 m_2 \\ \dot{\gamma}_1 &= \frac{m_3\gamma_2}{I_3} - \frac{m_2\gamma_3}{I_2} \\ \dot{\gamma}_2 &= \frac{m_1\gamma_3}{I_1} - \frac{m_3\gamma_1}{I_3} \\ \dot{\gamma}_3 &= \frac{m_2\gamma_1}{I_2} - \frac{m_1\gamma_2}{I_1}. \end{aligned} \tag{2}$$

ii) The center of mass lies on the axis of symmetry in the body, i. e.,

$$\chi = (0, 0, 1)$$

and

$$I_1 = I_2.$$

In this case the equations of motion take the following form:

$$\begin{aligned} \dot{m}_1 &= a_1 m_2 m_3 + mgl\gamma_2 \\ \dot{m}_2 &= a_2 m_1 m_3 - mgl\gamma_1 \\ \dot{m}_3 &= 0 \\ \dot{\gamma}_1 &= \frac{m_3\gamma_2}{I_3} - \frac{m_2\gamma_3}{I_1} \\ \dot{\gamma}_2 &= \frac{m_1\gamma_3}{I_1} - \frac{m_3\gamma_1}{I_3} \\ \dot{\gamma}_3 &= \frac{m_2\gamma_1}{I_1} - \frac{m_1\gamma_2}{I_1}. \end{aligned} \tag{3}$$

This top is usually called **Lagrangian top**.

3. Stability Problem

In this section we shall discuss the stability problem for the heavy top dynamics. A long but straightforward computation, or using for example *Maple* algebra system, leads us to:

Theorem 1. *The equilibrium states of our system (2) are:*

$$\begin{aligned} e_1^{MN} &= (0, 0, M, 0, 0, N), \quad M, N \in \mathbb{R} \\ e_2^{MN} &= \left(M, 0, N, \frac{(I_3 - I_1)MN}{mglI_1I_3}, 0, \frac{N^2(I_3 - I_1)}{mglI_3^2} \right), \quad M, N \in \mathbb{R} \\ e_3^{MN} &= \left(0, M, N, 0, \frac{(I_3 - I_2)MN}{mglI_2I_3}, \frac{N^2(I_3 - I_2)}{mglI_3^2} \right), \quad M, N \in \mathbb{R}. \end{aligned}$$

We shall concentrate here only to the equilibrium states e_1^{MN} , where we shall distinguish three particular cases, namely:

- i) $e_M^+ = (0, 0, M, 0, 0, 1), \quad M \in \mathbb{R}$
- ii) $e_M^- = (0, 0, M, 0, 0, -1), \quad M \in \mathbb{R}$
- iii) $e_M^0 = (0, 0, M, 0, 0, 0), \quad M \in \mathbb{R}.$

Then we have:

Theorem 2. (Holm *et al*[1]) *In the particular case of the Lagrangian top the equilibrium state e_M^+ is spectrally stable if and only if:*

$$M^2 > 4mglI_1.$$

Theorem 3. *In the particular case of the Lagrangian top the equilibrium state e_M^- is always spectrally stable.*

Proof: The characteristic polynomial associated to the matrix of the linear part of our system (3) at the equilibrium e_M^- is given by:

$$\begin{aligned} p(\lambda) &= \lambda^2 \left[\lambda^4 + \left(2mgl \frac{1}{I_1} - 2 \frac{M^2}{I_1I_3} + \frac{M^2}{I_1^2} + 2 \frac{M^2}{I_3^2} \right) \lambda^2 \right. \\ &\quad \left. + \frac{(mgl)^2}{I_1} + 2mgl \frac{M^2}{I_3I_1^2} - 2mgl \frac{M^2}{I_1I_3^2} + \frac{M^4}{I_1^2I_3^2} - 2 \frac{M^4}{I_1I_3^3} + \frac{M^4}{I_3^4} \right]. \end{aligned}$$

Its roots are given by:

$$\lambda_{1,2} = 0, \quad \lambda_{3,4} = \pm \sqrt{\frac{a + \sqrt{\Delta}}{2b}}, \quad \lambda_{5,6} = \pm \sqrt{\frac{a - \sqrt{\Delta}}{2b}}$$

where

$$a = -2mglI_1I_3^2 - M^2(2I_1^2 + I_3^2 - 2I_1I_3) < 0$$

$$b = I_1^2I_3^2 > 0$$

$$\Delta = M^2I_3^2(I_3 - 2I_1)^2(M^2 + 4mglI_1) > 0.$$

Let us now observe that:

$$\Delta - a^2 = -4I_1^2(M^2(I_3 - I_1) + mglI_3^2)^2 < 0$$

so that

$$\begin{cases} (a - \sqrt{\Delta})(a + \sqrt{\Delta}) > 0 \\ (a - \sqrt{\Delta}) + (a + \sqrt{\Delta}) = 2a < 0 \end{cases}$$

and consequently we can conclude that:

$$a - \sqrt{\Delta} < 0, \quad \text{and} \quad a + \sqrt{\Delta} < 0.$$

Hence the roots $\lambda_{3,4,5,6}$ are purely imaginary. Moreover, $\lambda = 0$ is a simple root of the minimal polynomial and therefore it follows that the equilibrium state e_M^- is spectrally stable. \square

Theorem 4. *The equilibrium state e_M^0 is always spectrally stable.*

Proof: An easy computation shows that the characteristic roots have the following values:

$$x_1 = x_2 = 0, \quad x_{3,4} = \pm i \frac{M}{I_3}, \quad x_{5,6} = \pm Mi\sqrt{-a_1a_2}$$

and then our assertion follows immediately via the expression for the minimal polynomial. \square

We shall now analyze the nonlinear stability of our equilibrium states e_M^+ , e_M^- , e_M^0 . More exactly we have:

Theorem 5. (Puta and Căruntu [3]) *If*

$$M^2 > 4mglI_1$$

then the equilibrium state e_M^+ is nonlinearly stable.

Remark 1. *In the particular case of the Lagrangian top we refined the result of Holm et al [1].*

Theorem 6. *If*

$$M^2 < mglI_3^2/(I_1 - I_3)$$

then the equilibrium state e_M^- is nonlinearly stable.

Proof: We shall use the energy-Casimir method. Let $H_\varphi \in C^\infty(\mathbb{R}^6, \mathbb{R})$ be the energy-Casimir function given by:

$$H_\varphi(m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3) = \frac{1}{2} \left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3} \right) + gl\gamma_3 \\ + \varphi \left(m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3, \frac{1}{2}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) \right)$$

where $\varphi \in C^\infty(\mathbb{R}^2, \mathbb{R})$.

Then the first variation of H_φ is given by:

$$DH_\varphi = \frac{m_1}{I_1} \delta m_1 + \frac{m_2}{I_2} \delta m_2 + \frac{m_3}{I_3} \delta m_3 + mgl\delta\gamma_3 \\ + \dot{\varphi}(m_1\delta\gamma_1 + m_2\delta\gamma_2 + m_3\delta\gamma_3 + \gamma_1\delta m_1 + \gamma_2\delta m_2 + \gamma_3\delta m_3) \\ + \varphi'(\gamma_1\delta\gamma_1 + \gamma_2\delta\gamma_2 + \gamma_3\delta\gamma_3).$$

At the equilibrium of interest we have:

$$DH_\varphi(e_M^-) = 0$$

if and only if:

$$\dot{\varphi} \left(-M, \frac{1}{2} \right) = \frac{M}{I_3} \quad \text{and} \quad \varphi' \left(-M, \frac{1}{2} \right) = mgl + \frac{M^2}{I_3}$$

where

$$\dot{\varphi} = \frac{\partial\varphi}{\partial(m \cdot \gamma)} \quad \text{and} \quad \varphi' = \frac{\partial\varphi}{\partial \left(\frac{1}{2} \|\gamma\|^2 \right)}.$$

Now, the second derivative of H_φ at the equilibrium of interest has the matrix:

$$\begin{pmatrix} \frac{1}{I_1} & 0 & 0 & \frac{M}{I_3} & 0 & 0 \\ 0 & \frac{1}{I_2} & 0 & 0 & \frac{M}{I_3} & 0 \\ 0 & 0 & \frac{1}{I_3} + A & 0 & 0 & -AM + B + \frac{M}{I_3} \\ \frac{M}{I_3} & 0 & 0 & mgl + \frac{M^2}{I_3} & 0 & 0 \\ 0 & \frac{M}{I_3} & 0 & 0 & mgl + \frac{M^2}{I_3} & 0 \\ 0 & 0 & -AM + B + \frac{M}{I_3} & 0 & 0 & 0 \end{pmatrix}$$

where

$$A = \ddot{\varphi} \left(-M, \frac{1}{2} \right), \quad B = \dot{\varphi}' \left(-M, \frac{1}{2} \right)$$

$$C = mgl + \frac{M^2}{I_3} + \varphi'' \left(-M, \frac{1}{2} \right) + M^2 A - 2MB.$$

Then the principal determinants are given by:

$$\frac{1}{I_1},$$

$$\frac{1}{I_1 I_2},$$

$$\frac{1 + AI_3}{I_1 I_2 I_3},$$

$$\frac{(1 + AI_3)[mglI_3^2 + M^2(I_3 - I_1)]}{I_1 I_2 I_3^3},$$

$$\frac{(1 + AI_3)[mglI_3^2 + M^2(I_3 - I_1)][mglI_3^2 + M^2(I_3 - I_2)]}{I_1 I_2 I_3^5},$$

$$\frac{[mglI_3^2 + M^2(I_3 - I_1)][mglI_3^2 + M^2(I_3 - I_2)]}{I_1 I_2 I_3^5}$$

$$\times \left[\frac{4M^2 A - 4MB + \varphi'' \left(-M, \frac{1}{2} \right) + mgl + A\varphi'' \left(-M, \frac{1}{2} \right) I_3}{I_1 I_2 I_3^5} \right.$$

$$\left. + \frac{AmglI_3 - B^2 I_3}{I_1 I_2 I_3^5} \right].$$

If we choose now φ such that:

$$\dot{\varphi} \left(-M, \frac{1}{2} \right) = \frac{M}{I_3}, \quad \varphi' \left(-M, \frac{1}{2} \right) = mgl + \frac{M^2}{I_3},$$

$$\ddot{\varphi} \left(-M, \frac{1}{2} \right) = 0, \quad \dot{\varphi}' \left(-M, \frac{1}{2} \right) = 0, \quad \varphi'' \left(-M, \frac{1}{2} \right) = 0$$

then all the principal determinants are positive and so e_M^- is nonlinearly stable. For instance, such a φ is given by:

$$\varphi(x, y) = \frac{M}{I_3} x + \left(mgl + \frac{M^2}{I_3} \right) y.$$

□

Remark 2. *It is an open problem to decide the nonlinear stability or instability of the equilibrium state e_M^0 .*

4. Periodic Orbits in the Heavy Top Dynamics

Let $G_\varphi \in C^\infty(\mathbb{R}^6, \mathbb{R})$ be the real valued function given by:

$$\begin{aligned} G_\varphi(m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3) &= \frac{1}{2} \left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3} \right) - \frac{M^2}{2I_3} \\ &\quad + mgl(1 + \gamma_3) \\ &\quad + \varphi \left(m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3, \frac{1}{2}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) \right) \\ &\quad - \varphi \left(-M, \frac{1}{2} \right) \end{aligned}$$

where $\varphi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ and moreover the following conditions are satisfied:

$$\begin{aligned} \dot{\varphi} \left(-M, \frac{1}{2} \right) &= \frac{M}{I_3}, \\ \varphi' \left(-M, \frac{1}{2} \right) &= mgl + \frac{M^2}{I_3}, \\ \ddot{\varphi} \left(-M, \frac{1}{2} \right) &= 0, \\ \dot{\varphi}' \left(-M, \frac{1}{2} \right) &= 0, \\ \varphi'' \left(-M, \frac{1}{2} \right) &= 0 \end{aligned}$$

where

$$\dot{\varphi} = \frac{\partial \varphi}{\partial (m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3)}$$

and

$$\varphi' = \frac{\partial \varphi}{\partial \left(\frac{1}{2}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) \right)}.$$

Then it is easy to prove that:

- i) G_φ is an integral of the system (2)
- ii) $G_\varphi(e_M^-) = 0$
- iii) $dG_\varphi(e_M^-) = 0$
- iv) If $M^2 < mglI_3^2/(I_1 - I_3)$, then $D^2G_\varphi(e_M^-)$ is positive definite.

Using now the Moser theorem (see Moser [2]) we have:

Theorem 7. *Let $M \in \mathbb{R}$ be a real number such that:*

$$M^2 < mglI_3^2/(I_1 - I_3).$$

Then for each ε sufficiently small any integral surface:

$$G_\varphi(m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3) = \varepsilon^2$$

contains at least one periodic solution of (2) whose periods are close to those of the corresponding linear system.

Let $K_\varphi \in C^\infty(\mathbb{R}^6, \mathbb{R})$ be the real valued function given by:

$$\begin{aligned} K_\varphi(m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3) &= \frac{1}{2} \left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3} \right) - \frac{M^2}{2I_3} \\ &\quad + mgl(\gamma_3 - 1) \\ &\quad + \varphi \left(m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3, \frac{1}{2}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) \right) \\ &\quad - \varphi \left(M, \frac{1}{2} \right) \end{aligned}$$

where $\varphi \in C^\infty(\mathbb{R}^2, \mathbb{R})$ and moreover the following conditions are satisfied:

$$\begin{aligned} \dot{\varphi} \left(M, \frac{1}{2} \right) &= -\frac{M}{I_3} \\ \varphi' \left(M, \frac{1}{2} \right) &= \frac{M^2}{I_3} - mgl \\ \ddot{\varphi} \left(M, \frac{1}{2} \right) &= 0 \\ \dot{\varphi}' \left(M, \frac{1}{2} \right) &= 0 \\ \varphi'' \left(M, \frac{1}{2} \right) &= 0 \end{aligned}$$

where

$$\dot{\varphi} = \frac{\partial \varphi}{\partial (m_1\gamma_1 + m_2\gamma_2 + m_3\gamma_3)}$$

and

$$\varphi' = \frac{\partial \varphi}{\partial \left(\frac{1}{2}(\gamma_1^2 + \gamma_2^2 + \gamma_3^2) \right)}.$$

Then it is easy to prove that:

- i) K_φ is an integral of the system (2)
- ii) $K_\varphi(e_M^+) = 0$
- iii) $dK_\varphi(e_M^+) = 0$
- iv) If $M^2 > 4mglI_1$, then $D^2K_\varphi(e_M^+)$ is positive definite.

Using again the Moser theorem (see Moser [2]) we have:

Theorem 8. *Let $M \in \mathbb{R}$ be a real number such that:*

$$M^2 > 4mglI_1.$$

Then for each ε sufficiently small any integral surface:

$$K_\varphi(m_1, m_2, m_3, \gamma_1, \gamma_2, \gamma_3) = \varepsilon^2$$

contains at least one periodic solution of (2) whose periods are close to those of the corresponding linear system.

References

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