

## ENUMERATIVE GEOMETRY FOR COMPLEX GEODESICS ON QUASI-HYPERBOLIC 4-SPACES WITH CUSPS

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**Abstract.** We introduce orbital functionals  $\int \beta$  simultaneously for each commensurability class of orbital surfaces. They are realized on infinitely dimensional *orbital* divisor spaces spanned by (arithmetic-geodesic real 2-dimensional) orbital curves on any orbital surface. We discover infinitely many of them on each commensurability class of orbital Picard surfaces, which are real 4-spaces with cusps and negative constant Kähler–Einstein metric degenerated along an orbital cycle. For a suitable (Heegner) sequence  $\int \mathbf{h}_N$ ,  $N \in \mathbb{N}$ , of them we investigate the corresponding formal orbital  $q$ -series  $\sum_{N=0}^{\infty} (\int \mathbf{h}_N) q^N$ . We show that after substitution  $q = e^{2\pi i \tau}$  and application to arithmetic orbital curves  $\hat{C}$  on a fixed Picard surface class, the series  $\sum_{N=0}^{\infty} (\int_{\hat{C}} \mathbf{h}_N) e^{2\pi i \tau}$  define modular forms of well-determined fixed weight, level and Nebentypus. The proof needs a new orbital understanding of orbital heights introduced in [12] and Mumford–Fulton’s rational intersection theory on singular surfaces in Riemann–Roch–Hirzebruch style. It has to be connected with Zeta and Theta functions of hermitian lines, indefinite quaternionic fields and of a matrix algebra along a research marathon over 75 years represented by Cogdell, Kudla, Hirzebruch, Zagier, Shimura, Schoeneberg and Hecke. Our aim is to open a door to an effective enumerative geometry for complex geodesics on orbital varieties with nice metrics.

### 1. Introduction

In the monograph [12] we defined orbital heights for orbital curves on orbital surfaces. In the most important cases of orbital hyperbolic surfaces (Picard surfaces), which are real 4-dimensional with cusps with negative constant cur-