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CURVATURE PROPERTIES OF SOME THREE-DIMENSIONAL ALMOST CONTACT MANIFOLDS WITH B-METRIC II

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Abstract. The curvature tensor on a 3-dimensional almost contact manifold with B-metric belonging to two main classes is studied. These classes are the rest of the main classes which were not considered in the first part of this work. The dimension 3 is the lowest possible dimension for the almost contact manifolds with B-metric. The corresponding curvatures are found and the respective geometric characteristics of the considered manifolds are given.

1. Preliminaries

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional almost contact manifold with B-metric, i.e. (φ, ξ, η) is an almost contact structure and g is a metric on M such that:

$$\varphi^2 = -\mathrm{id} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad g(\varphi X, \varphi Y) = -g(X,Y) + \eta(X)\eta(Y)$$

where $X, Y \in \mathcal{X}M$.

Both metrics g and its associated $\tilde{g}(X,Y) = g^*(X,Y) + \eta(X)\eta(Y)$ are indefinite metrics of signature $(n, n+1)$ [1], where it is denoted $g^*(X,Y) = g(X, \varphi Y)$.

Further, X, Y, Z, W will stand for arbitrary differentiable vector fields on M (i.e. the elements of $\mathcal{X}M$) and x, y, z, w are arbitrary vectors in the tangential space $T_pM, p \in M$.

Let $(V^{2n+1}, \varphi, \xi, \eta, g)$ be a $(2n+1)$-dimensional vector space with almost contact structure (φ, ξ, η) and B-metric g. It is well known the orthogonal decomposition $V = hV \oplus vV$ of $(V^{2n+1}, \varphi, \xi, \eta, g)$, where $hV = \{x \in V;\ x = hx = -\varphi^2 x\}$, $vV = \{x \in V;\ x = vx = \eta(x)\xi\}$. Denoting the restrictions of g and φ on hV by the same letters, we obtain the $2n$-dimensional almost complex vector space

$\{hV, \varphi, g\}$ with a complex structure φ and B-metric g. Then for arbitrary $x \in V$ we have $x = hx + \eta(x)\xi$. The basis $\{e_1, \dots, e_n, \varphi e_1, \dots, \varphi e_n, \xi\}$, where $-g(e_i, e_j) = g(\varphi e_i, \varphi e_j) = \delta_{ij}$, $g(e_i, \varphi e_j) = 0$, $\eta(e_i) = 0$, $i, j = 1, \dots, n$, is said to be an adapted φ-basis of V.

A decomposition of the class of the almost contact manifolds with B-metric with respect to the tensor $F : F(X, Y, Z) = g((\nabla_X \varphi)Y, Z)$ is given in [1], where eleven basic classes $\mathcal{F}_i$ $(i = 1, \dots, 11)$ are defined. The Levi-Civita connection of g is denoted by ∇. The special class $\mathcal{F}_0 : F = 0$ is contained in each of classes $\mathcal{F}_i$. The following 1-forms are associated with F: $\theta(x) = g^{ij}F(e_i, e_j, x)$, $\theta^*(x) = g^{ij}F(e_i, \varphi e_j, x)$, $\omega(x) = F(\xi, \xi, x)$, where $\{e_i, \xi\}$ $(i = 1, \dots, 2n)$ is a basis of T_pM and (g^{ij}) is the inverse matrix of (g_{ij}).

In this paper we consider two of the main classes engendered by the main components of F :

$$
\begin{aligned}
\mathcal{F}_1 :\ F(x, y, z) =& \frac{1}{2n}\{g(x, \varphi y)\theta(\varphi z) + g(x, \varphi z)\theta(\varphi y) + g(\varphi x, \varphi y)\theta(\varphi^2 z) \\
&+ g(\varphi x, \varphi z)\theta(\varphi^2 y)\} \\
\mathcal{F}_{11} : F(x, y, z) =& \eta(x)\{\eta(y)\omega(z) + \eta(z)\omega(y)\}.
\end{aligned}
$$

The subclasses $\mathcal{F}_1^0$, $\mathcal{F}_{11}^0$ are defined [2] by:

$$
\mathcal{F}_1^0 = \{M \in \mathcal{F}_1;\ \mathrm{d}\theta = \mathrm{d}\theta^* = 0\}, \quad \mathcal{F}_{11}^0 = \{M \in \mathcal{F}_{11};\ \mathrm{d}\omega \circ \varphi = 0\}.
$$

An almost contact manifold with B-metric in the class $\mathcal{F}_i$ we call an $\mathcal{F}_i$-manifold $(i = 0, 1, 2, \dots, 11)$ in short.

The curvature tensor R for ∇ is defined as ordinary by $R(X, Y, Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. The corresponding tensor of type $(0, 4)$ is denoted by the same letter and is given by $R(X, Y, Z, W) = g(R(X, Y, Z), W)$. The Ricci tensor ρ and the scalar curvature τ of R are given by $\rho(y, z) = g^{ij}R(e_i, y, z, e_j)$, $\tau = g^{ij}\rho(e_i, e_j)$, where $\{e_i\}$ $(i = 1, 2, \dots, 2n + 1)$ is a basis of T_pM.

A tensor L of type $(0, 4)$ is said to be a curvature-like tensor if it satisfies the conditions:

$$
L(X, Y, Z, W) = -L(Y, X, Z, W) = -L(X, Y, W, Z), \quad \underset{(X,Y,Z)}{\mathfrak{S}} L(X, Y, Z, W) = 0.
$$

A curvature-like tensor L is said to be a Kähler tensor if it satisfies the Kähler property $L(X, Y, Z, W) = -L(X, Y, \varphi Z, \varphi W)$.

Let S be a tensor of type $(0,2)$. We use the following tensors, invariant under the action of the structural group $(GL(n,\mathbb{C})\cap O(n,n))\times I$:

$$\begin{aligned}
\psi_1(S)(x,y,z,w) &= g(y,z)S(x,w) - g(x,z)S(y,w) + g(x,w)S(y,z)\\
&\quad - g(y,w)S(x,z)\\
\psi_2(S)(x,y,z,w) &= \psi_1(S)(x,y,\varphi z,\varphi w)\\
\psi_3(S)(x,y,z,w) &= -\psi_1(S)(x,y,\varphi z,w) - \psi_1(S)(x,y,z,\varphi w)\\
\psi_4(S)(x,y,z,w) &= \psi_1(S)(x,y,\xi,w)\eta(z) + \psi_1(S)(x,y,z,\xi)\eta(w)\\
\psi_5(S)(x,y,z,w) &= \psi_1(S)(x,y,\xi,\varphi w)\eta(z) + \psi_1(S)(x,y,\varphi z,\xi)\eta(w).
\end{aligned}$$

It is well known, that the tensors $\pi_i = \frac{1}{2}\psi_i(g)$ $(i = 1,2,3)$, $\pi_i = \psi_i(g)$ $(i = 4,5)$ are curvature-like tensors and $\pi_1 - \pi_2 - \pi_4$, $\pi_3 + \pi_5$ are Kähler tensors.

A decomposition of the space of curvature tensors $\mathcal{R}$ over $(V^{2n+1},\varphi,\xi,\eta,g)$ into 20 mutually orthogonal and invariant under the action of the structural group factors is obtained in [6]. It is valid the partial decomposition $\mathcal{R} = h\mathcal{R}\oplus v\mathcal{R}\oplus w\mathcal{R}$, where $h\mathcal{R} = \omega_1\oplus\cdots\oplus\omega_{11}$, $v\mathcal{R} = v_1\oplus\cdots\oplus v_5$, $w\mathcal{R} = w_1\oplus\cdots\oplus w_4$. The characteristic conditions of the factors ω_i $(i = 1,\dots,11)$, v_j $(j = 1,\dots,5)$ w_k $(k = 1,\dots,4)$ are given in [6]. Following [7], an almost contact manifold with B-metric is said to be in one of the classes ω_i, v_j, w_k if R belongs to the corresponding component.

Let (M^3,φ,ξ,η,g) be a 3-dimensional almost contact manifold with B-metric. According to [1] the class of these manifolds is $\mathcal{F}_1\oplus\mathcal{F}_4\oplus\mathcal{F}_5\oplus\mathcal{F}_8\oplus\mathcal{F}_9\oplus\mathcal{F}_{10}\oplus\mathcal{F}_{11}$. From the decomposition of $\mathcal{R}$ it follows that a 3-dimensional almost contact manifold with B-metric cannot belong to the factors ω_i $(i = 1,2,3,4,9,10,11)$, v_j $(j = 4,5)$.

Let us recall that we have

Proposition 1.1 ([4]). *The curvature tensor on every 3-dimensional almost contact manifold with B-metric has the form* $R = \psi_1(\rho) - \frac{\tau}{2}\pi_1$.

Proposition 1.2 ([4]). *Every 3-dimensional almost contact manifold with B-metric belongs to the class* $\omega_5\oplus v_1\oplus w\mathcal{R}$.

Lemma 1.1 ([4]). *Every Kähler curvature-like tensor on a 3-dimensional almost contact manifold with B-metric is zero.*

The curvature properties of a 3-dimensional $\mathcal{F}_i^0$-manifold $(i = 4,5)$ are studied in [4]. In this paper we consider analogous problems for a 3-dimensional $\mathcal{F}_i$-manifold $(i = 1,11)$. The present work completes the above mentioned investigations on the main classes of the considered manifolds. The curvature tensor identities for $\mathcal{F}_i^0$-manifold $(i = 1,11)$ are found in [3]. It is not difficult to verify that these identities are valid for the classes $\mathcal{F}_i$ $(i = 1,11)$, too.

2. Curvature Properties on a 3-dimensional $\mathcal{F}_1$-manifold

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an $\mathcal{F}_1$-manifold. Then its curvature tensor R satisfies the properties:

$$R(x, y, \xi) = 0 \tag{1}$$

$$\begin{aligned} R(x, y, \varphi z, \varphi w) = &-R(x, y, z, w) - \left\{ \frac{1}{2n}\{\psi_1 + \psi_2 - \psi_4\}(H) \right. \\ &\left. -\frac{1}{8n^2}\{\psi_1 + \psi_2 - \psi_4\}(P) - \frac{\theta(Q)}{4n^2}\{\pi_1 + \pi_2 - \pi_4\} \right\} (x, y, z, w) \end{aligned} \tag{2}$$

where

$$\begin{aligned} H(y, z) &= -(\nabla_y \theta)\varphi z - \frac{1}{4n}\{\theta(y)\theta(z) - \theta(\varphi y)\theta(\varphi z)\} \\ &= (\nabla_y \theta^*)z - \frac{1}{2n}\{\theta(Q)g(\varphi y, \varphi z) + \theta^*(Q)g(y, \varphi z)\} \\ &\quad + \frac{1}{4n}\{\theta(y)\theta(z) + 3\theta^*(y)\theta^*(z)\} \end{aligned}$$

and Q is the corresponding vector field of θ with respect to g, i.e. $\theta = g(Q, \cdot)$

$$P(y, z) = \theta(y)\theta(z) + \theta(\varphi y)\theta(\varphi z).$$

From (1) it follows $\rho(y, \xi) = \rho(\xi, y) = 0$. Obviously for the tensor fields H and P we have

$$H(y, \xi) = 0, \quad \operatorname{Tr} H = \operatorname{Tr}(\nabla\theta^*) + \frac{1}{2}\theta(Q), \quad \operatorname{Tr} H^* = \operatorname{Tr}(\nabla\theta) + \frac{1}{2}\theta^*(Q) \tag{3}$$

where $H^*(y, z) = H(y, \varphi z)$;

$$\begin{aligned} P(y, z) = P(z, y), \quad P(\varphi y, \varphi z) = P(y, z), \quad P(y, \xi) = P(\xi, y) = 0 \\ \operatorname{Tr} P = \operatorname{Tr} P^* = 0 \end{aligned} \tag{4}$$

where $P^*(y, z) = P(y, \varphi z)$.

Remark 2.1. *If $(M^{2n+1}, \varphi, \xi, \eta, g) \in \mathcal{F}_1^0$, then both 1-forms θ, θ^* are closed and consequently the tensor field H has the properties: $H(y, z) = H(z, y)$, $H(\varphi y, \varphi z) = -H(y, z)$* [3].

Lemma 2.1. *Let $(M^3, \varphi, \xi, \eta, g)$ be an $\mathcal{F}_1$-manifold. Then $\psi_1(P) = \psi_4(P)$ and $\psi_2(P) = 0$.*

Proof: Let $\{e_1, \varphi e_1, \xi\}$ be a φ-basis of T_pM, $p \in M$. For arbitrary $x \in T_pM$ we have the decomposition $x = x^1 e_1 + x^2 \varphi e_1 + \eta(x)\xi$. Taking into account (4) by direct computations we obtain immediately $\psi_1(P) = \psi_4(P)$ and $\psi_2(P) = 0$.

From Lemma 1.1 it follows that the Kähler tensor $\pi_1 - \pi_2 - \pi_4$ on $(M^3, \varphi, \xi, \eta, g)$ is zero. Using (2), Lemma 2.1 and $\pi_1 - \pi_2 - \pi_4 = 0$ for the curvature tensor of a 3-dimensional $\mathcal{F}_1$-manifold we have

$$R(x, y, \xi) = 0 \tag{5}$$

$$R(x, y, \varphi z, \varphi w) = -\left\{R + \frac{1}{2}\{\psi_1 + \psi_2 - \psi_4\}(H) - \frac{\theta(Q)}{2}\pi_2\right\}(x, y, z, w).$$

Proposition 1.1 and the last equality imply

$$\psi_1(\rho) + \psi_2(\rho) = -\frac{1}{2}\{\psi_1 + \psi_2 - \psi_4\}(H) + \frac{\tau}{2}\pi_1 + \frac{1}{2}\{\tau + \theta(Q)\}\pi_2. \tag{6}$$

After a contraction of (6) we obtain

$$\begin{aligned} 2\rho(y, z) &= \rho(\varphi y, \varphi z) - \frac{1}{2}\{\tau + \theta(Q) - \operatorname{Tr} H\}g(\varphi y, \varphi z) \\ &- \frac{1}{2}\{2\tau'' + \operatorname{Tr} H^*\}g(y, \varphi z) + \frac{1}{2}\{H(\varphi y, \varphi z) - H(y, z) - \eta(y)H(\xi, z)\} \end{aligned} \tag{7}$$

where $\tau'' = g^{ij}\rho(e_i, \varphi e_j)$.

By the substitution $y = \xi$ in (7) we find $H(\xi, z) = 0$. Having in mind $H(\xi, z) = H(z, \xi) = 0$ and the decomposition $x = x^1 e_1 + x^2 \varphi e_1 + \eta(x)\xi$ for arbitrary $x \in T_pM$ we establish the truthfulness of the following

Lemma 2.2. *Let $(M^3, \varphi, \xi, \eta, g)$ be an $\mathcal{F}_1$-manifold. Then we have:*

i) $\psi_2(H) = \operatorname{Tr} H\pi_2$;

ii) $\psi_1(H) = \psi_4(H) + \operatorname{Tr} H\pi_2$;

iii) $H(\varphi y, \varphi z) - H(y, z) = \operatorname{Tr} H g(\varphi y, \varphi z) + \operatorname{Tr} H^* g(y, \varphi z)$.

The property iii) from Lemma 2.2 implies $H(\varphi y, \varphi z) - H(y, z) = H(\varphi z, \varphi y) - H(z, y)$. In the last equality we substitute φz for z and using the definitions of H and $\mathrm{d}\theta$ ($\mathrm{d}\theta(y, z) = (\nabla_y\theta)z - (\nabla_z\theta)y$) we have

Corollary 2.1. *For every 3-dimensional $\mathcal{F}_1$-manifold we have $(\mathrm{d}\theta) \circ \varphi = \mathrm{d}\theta$.*

Theorem 2.1. *The curvature tensor, the Ricci tensor and the scalar curvature on a 3-dimensional $\mathcal{F}_1$-manifold are given respectively by:*

$$R(x, y, z, w) = \frac{\tau}{2}\pi_2(x, y, z, w) \tag{8}$$

$$\rho(y, z) = -\frac{\tau}{2}g(\varphi y, \varphi z) \tag{9}$$

$$\tau = -\operatorname{Tr} H + \frac{\theta(Q)}{2} = -\operatorname{Tr}(\nabla\theta^*). \tag{10}$$

Proof: Taking into account the equalities i) and ii) from Lemma 2.2, the equality (6) gets the form

$$\psi_1(\rho) + \psi_2(\rho) = \frac{\tau}{2}\pi_1 + \frac{1}{2}\{\tau + \theta(Q) - 2\operatorname{Tr} H\}\pi_2. \tag{11}$$

After the substitution $y = w = \xi$ in (11) and because of $\rho(\xi, z) = 0$ we obtain (9). Then Proposition 1.1 and (9) imply (8). Finally, using (9) and (11) we compute the scalar curvature τ of R.

The equality iii) of Lemma 2.2 and Remark 2.1 imply the following form of the tensor H on a 3-dimensional $\mathcal{F}_1^0$-manifold

$$H(y, z) = -\frac{1}{2}\{\operatorname{Tr} H g(\varphi y, \varphi z) + \operatorname{Tr} H^* g(y, \varphi z)\}.$$

3. Curvature Properties on a 3-dimensional $\mathcal{F}_{11}$-manifold

Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an $\mathcal{F}_{11}$-manifold. Then the curvature tensor R on $(M^{2n+1}, \varphi, \xi, \eta, g)$ satisfies the properties:

$$R(x, y, \xi) = \psi_4(S_{11})(x, y, \xi) \tag{12}$$

$$R(x, y, \varphi z, \varphi w) = -R(x, y, z, w) + \psi_4(S_{11})(x, y, z, w) \tag{13}$$

where

$$S_{11}(y, z) = (\nabla_y \omega)\varphi z - \omega(\varphi y)\omega(\varphi z) + \eta(y)\eta(z)\omega(\Omega) = (\nabla_y \tilde{\omega})z - \tilde{\omega}(y)\tilde{\omega}(z)$$
$$\tilde{\omega} = \omega \circ \varphi$$

and Ω is the corresponding vector field of ω with respect to g, i.e. $\omega = g(\Omega, \cdot)$.

From (12) it follows $\rho(y, \xi) = \rho(\xi, y) = \eta(y)\operatorname{Tr}(\nabla\tilde{\omega})$ and for the tensor field S_{11} we have

$$S_{11}(\xi, y) = (\nabla_\xi \omega)\varphi y + \eta(y)\omega(\Omega),\ S_{11}(y, \xi) = \eta(y)\omega(\Omega),\ S_{11}(\xi, \xi) = \omega(\Omega)$$
$$\operatorname{Tr} S_{11} = \operatorname{Tr}(\nabla\tilde{\omega}) + \omega(\Omega), \quad \operatorname{Tr} S_{11}^* = -\operatorname{Tr}(\nabla\omega)$$

where $S_{11}^*(y, z) = S_{11}(y, \varphi z)$.

Remark 3.1 ([3]). *If $(M^3, \varphi, \xi, \eta, g) \in \mathcal{F}_{11}^0$, then the 1-form $\omega \circ \varphi$ is closed and consequently the tensor field S_{11} is symmetric.*

Let $(M^3, \varphi, \xi, \eta, g)$ be an $\mathcal{F}_{11}$-manifold. Then from Proposition 1.1 and (13) we obtain

$$\{\psi_1 + \psi_2\}(\rho) = \frac{\tau}{2}\{\pi_1 + \pi_2\} + \psi_4(S_{11}) \tag{14}$$

After two contractions of (14) we find the following two equalities:

$$\begin{aligned}2\rho(y,z) &= \rho(\varphi y,\varphi z) - \tau'' g(y,\varphi z) - \frac{\tau}{2} g(\varphi y,\varphi z)\\ &\quad + 2\operatorname{Tr}(\nabla\tilde{\omega})\eta(y)\eta(z) + S_{11}(y,z) - \eta(y)S_{11}(\xi,z) \quad (15)\end{aligned}$$

$$\rho(y,\varphi z) + \rho(\varphi y,z) = \tau'' g(y,z) - \operatorname{Tr} S^*_{11}\eta(y)\eta(z) + \operatorname{Tr}(\nabla\tilde{\omega})g(y,\varphi z). \quad (16)$$

From (16) we compute $\tau'' = \operatorname{Tr} S^*_{11}$. Substituting τ'' and $y = \varphi y$ in (16) we have

$$\rho(\varphi y,\varphi z) = \rho(y,z) + \operatorname{Tr} S^*_{11} g(y,\varphi z) - \frac{\tau}{2} g(y,z) \quad (17)$$

Lemma 3.1. *Let (M^3,φ,ξ,η,g) be an $\mathcal{F}_{11}$-manifold. The tensors $\psi_1(S_{11})$ and $\psi_4(S_{11})$ are related as follows*

$$\begin{aligned}\psi_1(S_{11})(x,y,z,w) &= \psi_4(S_{11})(x,y,z,w) + \psi_1(S_{11}(\eta\otimes\xi,\cdot))(x,y,z,w)\\ &\quad + \operatorname{Tr}(\nabla\tilde{\omega})\pi_2(x,y,z,w).\end{aligned} \quad (18)$$

The proof is a straightforward calculation using formula (3).

Theorem 3.1. *The curvature tensor, the Ricci tensor and the scalar curvature on a 3-dimensional $\mathcal{F}_{11}$-manifold are,*

$$R(x,y,z,w) = \psi_4(S_{11})(x,y,z,w) \quad (19)$$

$$\rho(y,z) = hS_{11}(y,z) + \frac{\tau}{2}\eta(y)\eta(z) \quad (20)$$

respectively, where

$$hS_{11}(y,z) = S_{11}(hy,hz), \qquad \tau = 2\operatorname{Tr}(\nabla\tilde{\omega}). \quad (21)$$

Proof: From (15) and (17) we find $\tau = 2\operatorname{Tr}(\nabla\tilde{\omega})$ and

$$\rho(y,z) = S_{11}(y,z) - \eta(y)S_{11}(\xi,z) + \frac{\tau}{2}\eta(y)\eta(z). \quad (22)$$

For arbitrary $x \in T_pM$ we have $x = hx + \eta(x)\xi$ and it is easy to check $S_{11}(y,z) - \eta(y)S_{11}(\xi,z) = hS_{11}(y,z)$. From the last equality and (22) we obtain (20). Finally, Proposition 1.1, Lemma 3.1 and (20) imply (19).

Because of $\rho(y,z) = \rho(z,y)$ and (20) it is valid the following

Proposition 3.1. *For every 3-dimensional $\mathcal{F}_{11}$-manifold we have*

$$hS_{11}(y,z) = hS_{11}(z,y).$$

The statement of the last proposition implies immediately

Corollary 3.1. *The 1-form ω of a 3-dimensional $\mathcal{F}_{11}$-manifold satisfies the following equality*

$$(\nabla_{\varphi^2 y}\omega)\varphi z = (\nabla_{\varphi^2 z}\omega)\varphi y.$$

4. Geometric Characteristics of the 3-dimensional $\mathcal{F}_i$-manifolds $(i = 1, 11)$

According to the decomposition of $\mathcal{R}$ [6], from Theorem 2.1 and Theorem 3.1 we have

Proposition 4.1. *The class of the 3-dimensional $\mathcal{F}_i$-manifolds for $i = 1$ and $i = 11$ is ω_5 and $w\mathcal{R}$, respectively.*

Let us recall from [4] that an almost contact manifold with B-metric is said to be a φ-Einstein manifold, or a v-Einstein manifold if $\rho = -\alpha g(\varphi\cdot, \varphi\cdot)$, $\rho = \gamma\eta \otimes \eta$ $(\alpha, \gamma \neq \mathrm{const})$, respectively.

Having in mind the form of the Ricci tensor from Theorem 2.1 and Theorem 3.1, the following propositions are valid

Proposition 4.2. *A 3-dimensional $\mathcal{F}_1$-manifold is φ-Einstein iff* $\mathrm{Tr}(\nabla\theta^*) = \mathrm{const}$.

Proposition 4.3. *A 3-dimensional $\mathcal{F}_{11}$-manifold is v-Einstein iff $hS_{11} = 0$ and* $\mathrm{Tr}(\nabla\tilde{\omega}) = \mathrm{const}$.

The sectional curvature $K(x, y) = \dfrac{R(x, y, y, x)}{\pi_1(x, y, y, x)}$ with respect to g and R for every nondegenerate section α with a basis $\{x, y\}$ in T_pM is known. The following special sections in T_pM, $\dim M = 2n + 1$: a ξ-section (i.e. $\{\xi, x\}$), a φ-holomorphic section (i.e. $\alpha = \varphi\alpha$) and a totally real section (i.e. $\alpha \perp \varphi\alpha$) are introduced in [5]. Note that totally real sections do not exist in the 3-dimensional case.

Using Theorem 2.1 and Theorem 3.1 we compute the sectional curvatures of a ξ-section and a φ-holomorphic section on a 3-dimensional $\mathcal{F}_i$-manifold $(i = 1, 11)$:

- $i = 1$

$$K(\xi, x) = 0, \quad K(\varphi x, \varphi^2 x) = \frac{\tau}{2} = -\frac{\mathrm{Tr}(\nabla\theta^*)}{2} \tag{23}$$

- $i = 11$

$$K(\xi, x) = -\frac{S_{11}(hx, hy)}{g(\varphi x, \varphi y)}, \quad K(\varphi x, \varphi^2 x) = 0. \tag{24}$$

Formulas (23) and (24) imply

Proposition 4.4. *Let $(M^3, \varphi, \xi, \eta, g)$ be an $\mathcal{F}_i$-manifold ($i = 1, 11$). Then we have:*

- $i = 1$
  i) *The sectional curvatures of the ξ-sections are zero*
  ii) *M has constant φ-holomorphic sectional curvatures iff M is a φ-Einstein manifold*

- $i = 11$

  iii) *The φ-holomorphic sectional curvatures are zero*

  iv) *The sectional curvatures of the ξ-sections are zero iff M is a v-Einstein manifold.*

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