

THE HYPERBOLIC TRIANGLE DEFECT

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Abstract. The hyperbolic trigonometry, fully analogous to the common Euclidean trigonometry, is presented and employed to calculate the hyperbolic triangle defect in the Poincaré ball model of n -dimensional hyperbolic geometry. It is shown that hyperbolic trigonometry allows the hyperbolic triangle defect to be expressed in terms of the triangle hyperbolic side lengths by a remarkably elegant identity.

1. Introduction

Gyrovector spaces form the setting for hyperbolic geometry and its hyperbolic trigonometry just as vector spaces form the setting for Euclidean hyperbolic geometry and its Euclidean trigonometry. Gyrovector spaces, in turn, are gyrocommutative gyrogroups that admit scalar multiplication just as vector spaces are commutative groups that admit scalar multiplication.

Accordingly, to set the stage for the introduction of hyperbolic geometry and its applications, the definitions of abstract gyrogroups and gyrovector spaces are presented in Section 2. A concrete example of a family of n -dimensional gyrovector spaces (n finite or infinite), called the Möbius gyrovector spaces, is presented in Section 3. It is then demonstrated in Section 4 that Möbius gyrovector spaces form the setting for the Poincaré ball model of hyperbolic geometry just as vector spaces form the setting for the standard model of Euclidean geometry. Having the stage set, the hyperbolic geometry of the Poincaré ball model of hyperbolic geometry is presented in Section 5 including, in particular,

- 1) the Hyperbolic Law of Cosines and its resulting Hyperbolic Pythagorean Theorem that shares visual analogies with its Euclidean counterpart, Fig. 2, and
- 2) the Hyperbolic Law of Sines.