

BREATHER SOLUTIONS OF N -WAVE EQUATIONS

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Abstract. We consider N -wave type equations related to symplectic and orthogonal algebras. We obtain their soliton solutions in the case when two different \mathbb{Z}_2 reductions (or equivalently one $\mathbb{Z}_2 \times \mathbb{Z}_2$ -reduction) are imposed. For that purpose we apply a particular case of an auto-Bäcklund transformation – the Zakharov–Shabat dressing method. The corresponding dressing factor is consistent with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ -reduction. These soliton solutions represent N -wave breather-like solitons. The discrete eigenvalues of the Lax operators connected with these solitons form “quadruplets” of points which are symmetrically situated with respect to the coordinate axes.

1. Introduction

The N -wave equation related to a semisimple Lie algebra \mathfrak{g} is a matrix system of nonlinear differential equations of the type

$$i[J, Q_t(x, t)] - i[I, Q_x(x, t)] + [[I, Q(x, t)], [J, Q(x, t)]] = 0 \quad (1)$$

where the squared brackets denote the commutator of matrices and the subscript means a partial derivative with the respect to independent variables t and x . The constant matrices I and J are regular elements of the Cartan subalgebra \mathfrak{h} of the Lie algebra \mathfrak{g} . The matrix-valued function $Q(x, t) \in \mathfrak{g}$ can be expanded as follows

$$Q = \sum_{\alpha \in \Delta} Q_\alpha(x, t) E_\alpha$$

where Δ denotes the root system of \mathfrak{g} and E_α are elements of Weyl basis of Lie algebra \mathfrak{g} parametrized by roots of \mathfrak{g} . It is also assumed that $Q(x, t)$ satisfies a vanishing boundary condition, i.e., $\lim_{x \rightarrow \pm\infty} Q(x, t) = 0$.

The N -wave equation is an example of an S -integrable evolutionary equation. Such a type of equations appears in nonlinear optics and describes the propagation of N wave packets in nonlinear media (see [1]).

Another application of the N -wave systems is in the differential geometry. Ferapontov [2] showed that N -wave equations naturally occurred when one studied isoparametric hypersurfaces in spheres.

The problem for classification and investigation of all admissible reductions of an integrable equation is one of the fundamental problems in the theory of integrable systems. N -wave equations with canonical \mathbb{Z}_2 symmetries have been discussed for the first time by Zakharov and Manakov in [16] for $\mathfrak{g} \simeq \mathfrak{sl}(n, \mathbb{C})$. More recently, the \mathbb{Z}_2 -reductions of the N -wave equations related to the low-rank simple Lie algebras were analyzed [7] and classified [8]. Our aim in this paper is actually two-fold. First, we outline the derivation of the soliton solutions of the N -wave equations with \mathbb{Z}_2 -reductions. Second, we further reduce the N -wave equations by imposing a second \mathbb{Z}_2 -reduction. As a result we derive a special class of N -wave equations related to orthogonal and symplectic algebras which, like the sine-Gordon equation, possess breather type solutions. Next, we obtain the soliton solutions themselves applying one of the basic methods in theory of integrable systems — the Zakharov–Shabat dressing procedure. The additional symmetries of the nonlinear equations have been taken into account when we choose a proper dressing factor. The soliton solutions of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ -reduced N -wave equation are analogues of breather solutions of the well-known sine-Gordon equation — they are associated with four discrete eigenvalues of Lax operators situated symmetrically with respect to the coordinate frame in the complex plane of the spectral parameter. Since we shall deal with the inverse scattering transform we begin with a reminder of all necessary facts concerning that theory. For more detailed information we recommend the books [14, 17].

2. General Formalism

As we mentioned above the N -wave system (1) is an integrable one. It admits a Lax representation with Lax operators

$$\begin{aligned} L\psi(x, t, \lambda) &:= (i\partial_x + U(x, t, \lambda))\psi(x, t, \lambda) = 0 \\ M\psi(x, t, \lambda) &:= (i\partial_t + V(x, t, \lambda))\psi(x, t, \lambda) \end{aligned} \quad (2)$$

where λ is an auxiliary (so-called spectral) parameter and the potentials $U(x, t, \lambda)$, $V(x, t, \lambda)$ are elements of \mathfrak{g} which are linear functions on λ defined by

$$\begin{aligned} U(x, t, \lambda) &:= U^0(x, t) - \lambda J = [J, Q(x, t)] - \lambda J \\ V(x, t, \lambda) &:= V^0(x, t) - \lambda I = [I, Q(x, t)] - \lambda I. \end{aligned} \quad (3)$$

The nonlinear evolution equation itself is equivalent to the compatibility condition of the differential operators L and M

$$[L, M] = 0 \quad \Leftrightarrow \quad i[J, Q_t] - i[I, Q_x] + [[I, Q], [J, Q]] = 0.$$

Since L and M commute they have the same eigenfunctions and the system (2) can be presented by

$$\begin{aligned} L\psi(x, t, \lambda) &:= (i\partial_x + U(x, t, \lambda))\psi(x, t, \lambda) = 0 \\ M\psi(x, t, \lambda) &:= (i\partial_t + V(x, t, \lambda))\psi(x, t, \lambda) = \psi(x, t, \lambda)C(\lambda). \end{aligned} \quad (4)$$

where C is a constant matrix with respect to x and t . The fundamental solutions $\psi(x, t, \lambda)$ of the auxiliary linear system (4) take values in the Lie group G corresponding to the Lie algebra \mathfrak{g} .

In order to find the soliton solutions we need of the so-called **fundamental analytic solutions** (FAS). There is a standard algorithm to construct these solutions by using another class of fundamental solutions of the linear problem (4) – Jost solutions. The Jost solutions $\psi_{\pm}(x, t, \lambda)$ are determined by their asymptotics at infinity, i.e.,

$$\lim_{x \rightarrow \pm\infty} \psi_{\pm}(x, t, \lambda)e^{i\lambda Jx} = \mathbb{1}.$$

Remark. This definition is correct provided we have fixed the matrix-valued function C by

$$C(\lambda) := \lim_{x \rightarrow \pm\infty} V(x, t, \lambda) = -\lambda I.$$

i.e., the asymptotics of ψ_{\pm} are t -independent. $C(\lambda)$ is directly related to the dispersion law of the nonlinear equation. Thus, the dispersion law of the N -wave equation is a linear function of the spectral parameter λ .

The Jost solutions $\psi_{\pm}(x, t, \lambda)$ are linearly dependent which means that there exists a matrix $T(t, \lambda)$ such that

$$\psi_{-}(x, t, \lambda) = \psi_{+}(x, t, \lambda)T(t, \lambda).$$

The matrix-valued function T is called a scattering matrix. Its time evolution is determined by the second equation of (4), i.e.,

$$i\partial_t T(t, \lambda) - \lambda[I, T(t, \lambda)] = 0.$$

Consequently

$$T(t, \lambda) = e^{-i\lambda It}T(0, \lambda)e^{i\lambda It}. \quad (5)$$

The **inverse scattering transform** (IST) allows one to solve the Cauchy problem for the nonlinear evolution equation, i.e., finding a solution $Q(x, t)$ when its initial condition $Q_{\text{in}}(x) := Q(x, 0)$ is given. The idea of the IST is illustrated in the following diagram

$$Q_{\text{in}} \rightarrow U(x, 0, \lambda) \xrightarrow{\text{DSP}} T(0, \lambda) \longrightarrow T(t, \lambda) \xrightarrow{\text{ISP}} U(x, t, \lambda) \rightarrow Q(x, t).$$

The first step consists in constructing the scattering matrix at some initial moment $t = 0$ by using the potential at the same moment (or equivalently by the solution Q_{in} at that moment). This is the **direct scattering problem** (DSP). The evolution of the scattering matrix (data) is already known and it is given by (5). The third step is a recovering of the potential U and respectively the solution of the nonlinear equation Q at an arbitrary moment from the scattering data at that moment – this is the **inverse scattering problem** (ISP). That step is actually the only nontrivial one. Thus following all steps we can solve the Cauchy problem for the nonlinear evolution equation. Since we know the evolution of scattering data we can easily determine the time dependence of fundamental solutions, solutions of nonlinear problem, etc.

The Jost solutions are defined for real values of λ only (they do not possess analytic properties for $\lambda \notin \mathbb{R}$). For our purpose it is necessary to construct fundamental solutions which admit analytic continuation beyond the real axes. It can be shown that there exist fundamental solutions $\chi^+(x, t, \lambda)$ and $\chi^-(x, t, \lambda)$ analytic in the upper half-plane \mathbb{C}_+ and in the lower half-plane \mathbb{C}_- of the spectral parameter, respectively. They can be obtained from the Jost solutions by a simple algebraic procedure proposed by Shabat [13], see also [17]. The procedure uses a Gauss decomposition of the scattering matrix $T(t, \lambda)$, namely

$$\chi^\pm(x, t, \lambda) = \psi_\mp(x, t, \lambda)S^\pm(\lambda) = \psi_\pm(x, t, \lambda)T^\mp(t, \lambda)D^\pm(\lambda)$$

where the matrices S^\pm , T^\pm and D^\pm are Gauss factors of the matrix T , i.e.,

$$T(t, \lambda) = T^\mp(t, \lambda)D^\pm(\lambda)(S^\pm(t, \lambda))^{-1}.$$

The matrices $S^+(t, \lambda)$ (respectively $S^-(t, \lambda)$) and $T^+(t, \lambda)$ (respectively $T^-(t, \lambda)$) are upper (respectively lower) triangular with unit diagonal elements whose time dependence is given by

$$i\partial_t S^\pm(t, \lambda) - \lambda[I, S^\pm(t, \lambda)] = 0, \quad i\partial_t T^\pm(t, \lambda) - \lambda[I, T^\pm(t, \lambda)] = 0. \quad (6)$$

The matrices $D^+(\lambda)$ and $D^-(\lambda)$ are diagonal and allow analytic extension in λ for $\Im\lambda > 0$ and $\Im\lambda < 0$. They do not depend on time and actually they provide the generating functionals of the integrals of motion of the nonlinear evolution equation [4, 7, 8, 17], see also the review paper [5].

A powerful method for obtaining solutions to nonlinear differential equations is the Bäcklund transformation (see [10] and [12] for more detailed information). The Bäcklund transformation maps a solution of a given differential equation into a solution of another differential equation. If both equations coincide one speaks about auto-Bäcklund transformation. A very important particular case of an auto-Bäcklund transformation is the dressing method proposed by Zakharov and Shabat [18]. Its basic idea consists in constructing a new solution $Q(x, t)$ starting from a

known solution $Q_0(x, t)$ taking into account the existence of the auxiliary linear system (4).

Let $\psi_0(x, t, \lambda)$ satisfy the linear problem

$$L_0\psi_0 := i\partial_x\psi_0 + ([J, Q_0(x, t)] - \lambda J)\psi_0(x, t, \lambda) = 0. \quad (7)$$

We construct a function $\psi(x, t, \lambda)$ by introducing a gauge transformation $g(x, t, \lambda)$ – a dressing of the solution ψ_0

$$\psi_0(x, t, \lambda) \rightarrow \psi(x, t, \lambda) = g(x, t, \lambda)\psi_0(x, t, \lambda)$$

such that the linear system (7) is covariant under the action of that gauge transformation. Thus the dressing factor has to satisfy the equation

$$i\partial_x g + [J, Q(x, t)]g(x, t, \lambda) - g(x, t, \lambda)[J, Q_0(x, t)] - \lambda[J, g(x, t, \lambda)] = 0. \quad (8)$$

If we choose a dressing factor which is a meromorphic function of the spectral parameter λ as follows

$$g(x, t, \lambda) = \mathbb{1} + \frac{A(x, t)}{\lambda - \lambda^+} + \frac{B(x, t)}{\lambda - \lambda^-} \quad (9)$$

where $\lambda^\pm \in \mathbb{C}_\pm$, we obtain the following relation between Q and Q_0

$$[J, Q(x, t)] = [J, Q_0(x, t) + A(x, t) + B(x, t)].$$

As a result we are able to construct new solutions if we know the functions A and B . We will show later how this can be done.

The simplest class of solutions consists of the so-called reflectionless potentials. A soliton solution is obtained by dressing the trivial solution $Q_0 \equiv 0$. Then the fundamental solution of the linear problem is just a plane wave $\psi_0(x, \lambda) = e^{-i\lambda Jx}$ and the one-soliton solution itself is given by

$$[J, Q_{1s}(x, t)] = [J, A_{1s}(x, t) + B_{1s}(x, t)].$$

As we said above the dressing procedure maps a solution of the linear problem to another solution of a linear problem with a different potential. In particular the Jost solutions $\psi_{0,\pm}$ are transformed into

$$\psi_\pm(x, t, \lambda) = g(x, t, \lambda)\psi_{0\pm}(x, t, \lambda)g_\pm^{-1}(\lambda)$$

where the factors $g_\pm(\lambda)$ ensure the proper asymptotics of the dressed solutions and are defined by

$$g_\pm(\lambda) := \lim_{x \rightarrow \pm\infty} g(x, t, \lambda).$$

Hence the dressed scattering matrix T reads

$$T(t, \lambda) = g_+(\lambda)T_0(t, \lambda)g_-^{-1}(\lambda).$$

It can be proven that the FAS $\chi_0^\pm(x, t, \lambda)$ transform into

$$\chi^\pm(x, t, \lambda) = g(x, t, \lambda)\chi_0^\pm(x, t, \lambda)g_-^{-1}(\lambda). \quad (10)$$

The spectral properties of L are determined by the behaviour of its resolvent operator. The resolvent R is an integral operator (see [3, 5] for more details) given by

$$R(t, \lambda)f(x) = \int_{-\infty}^{\infty} \mathcal{R}(x, y, t, \lambda)f(y) dy$$

where the integral kernel $\mathcal{R}(x, y, t, \lambda)$ must be a piece-wise analytic function of λ satisfying the equation

$$L\mathcal{R}(x, y, t, \lambda) = \delta(x - y)\mathbb{1}.$$

The kernel \mathcal{R} can be constructed by using the fundamental analytical solutions as follows

$$\mathcal{R}(x, y, t, \lambda) = \begin{cases} \mathcal{R}^+(x, y, t, \lambda), & \Im(\lambda) > 0 \\ \mathcal{R}^-(x, y, t, \lambda), & \Im(\lambda) < 0 \end{cases}$$

where

$$\begin{aligned} \mathcal{R}^{\pm}(x, y, t, \lambda) &:= \pm i\chi^{\pm}(x, t, \lambda)\Theta^{\pm}(x - y)(\chi^{\pm}(y, t, \lambda))^{-1} \\ \Theta^{\pm}(x - y) &:= \theta(\mp(x - y))\Pi - \theta(\pm(x - y))(1 - \Pi) \\ \Pi &:= \sum_{p=1}^n E_{pp}, \quad (E_{pq})_{rs} := \delta_{pr}\delta_{qs}. \end{aligned}$$

Due to the fact that we have chosen J to be a real matrix with $J_1 > J_2 > \dots > J_n$ the resolvent $R(t, \lambda)$ is a bounded integral operator for $\Im\lambda \neq 0$. For $\Im\lambda = 0$ $R(t, \lambda)$ is an unbounded integral operator, which means that the continuous spectrum of L fills up the real axes \mathbb{R} . Since the discrete part of the spectrum of L is determined by the poles of R it coincides with the poles and zeroes of χ^{\pm} .

From (10) and from the explicit form of \mathcal{R} it follows that the dressed kernel is related to the bare one by

$$\mathcal{R}^{\pm}(x, y, t, \lambda) = g(x, t, \lambda)\mathcal{R}_0^{\pm}(x, y, t, \lambda)g^{-1}(y, t, \lambda). \quad (11)$$

If we assume that the “bare” operator L_0 has no discrete eigenvalues then the poles of g determine the discrete eigenvalues of L

$$L_0 \xrightarrow{g} L \quad \Leftrightarrow \quad \text{spec}(L_0) \rightarrow \text{spec}(L) = \text{spec}(L_0) \cup \{\lambda^+, \lambda^-\}.$$

Many classical integrable systems correspond to Lax operators with potentials possessing additional symmetries. That is why it is of particular interest to consider the case when certain symmetries are imposed on the potential U (respectively on the solution Q).

Let G_R be a discrete group acting in G by group automorphisms and in \mathbb{C} by the conformal transformations

$$\kappa : \lambda \rightarrow \kappa(\lambda).$$

Therefore, we have an induced action of G_R in the space of functions $f(x, t, \lambda)$ taking values in G as follows

$$\mathcal{K} : f(x, t, \lambda) \rightarrow \tilde{f}(x, t, \lambda) := \tilde{K} \left(f \left(x, t, \kappa^{-1}(\lambda) \right) \right), \quad \tilde{K} \in \text{Aut}(G).$$

This action in turn induces another action on the differential operators L and M

$$\tilde{L}(\lambda) = \mathcal{K}L(\kappa^{-1}(\lambda))\mathcal{K}^{-1}, \quad \tilde{M}(\lambda) = \mathcal{K}M(\kappa^{-1}(\lambda))\mathcal{K}^{-1}.$$

The group action is consistent with Lax representation which is equivalent to invariance of the Lax representation under the action of G_R , i.e.,

$$[\tilde{L}, \tilde{M}] = [L, M] = 0.$$

The requirement regarding the G_R -invariance of the set of fundamental solutions $\{\psi(x, t, \lambda)\}$ leads to the following symmetry condition

$$\tilde{K}U(x, t, \kappa^{-1}(\lambda))\tilde{K}^{-1} = U(x, t, \lambda). \quad (12)$$

In other words the potential U , as well as Q are reduced. This fact motivates the name of the group G_R – a reduction group. The concept of the reduction group was proposed by Mikhailov [11].

One can prove that the dressing factor g ought to be invariant under the action of the reduction group G_R

$$\tilde{K} \left(g(x, t, \kappa^{-1}(\lambda)) \right) = g(x, t, \lambda). \quad (13)$$

Example 1. Canonical \mathbb{Z}_2 reduction.

Consider the group \mathbb{Z}_2 acting on \mathbb{C} by the complex conjugation $\lambda \rightarrow \lambda^*$ and on the orthogonal (or symplectic) group by an inner automorphism

$$\mathcal{X} \rightarrow K_1 \left(\mathcal{X}^\dagger \right)^{-1} K_1^{-1}$$

where $\mathcal{X}, K_1 \in \text{SO}(n, \mathbb{C})$ (respectively $\mathcal{X}, K_1 \in \text{Sp}(2n, \mathbb{C})$) and \dagger stands for the hermitian conjugation. Typically K_1 is chosen as a \mathbb{Z}_2 element of the Cartan subgroup, i.e., a diagonal matrix with diagonal elements $s_l = \pm 1$. Of course, one can consider also reductions in which K_1 is given by a Weyl reflection [7, 8].

The induced action on $\chi^\pm(x, t, \lambda)$ is

$$\chi^+(x, t, \lambda) = K_1((\chi^-)^\dagger(x, t, \lambda^*))^{-1}K_1^{-1}$$

and the symmetry condition (12) now reads

$$K_1U^\dagger(x, t, \lambda^*)K_1^{-1} = U(x, t, \lambda) \quad \Rightarrow \quad Q(x, t) = -K_1Q^\dagger(x, t)K_1^{-1}.$$

Let us consider as a simple illustration the four-wave equation associated with $\mathfrak{so}(5)$ algebra. Then $K = \text{diag}(s_1, s_2, 1, s_2, s_1)$ and the matrix-valued function Q

has the form

$$Q = \begin{pmatrix} 0 & Q_{12} & Q_{13} & Q_{14} & 0 \\ -s_1 s_2 Q_{12}^* & 0 & Q_{23} & 0 & Q_{14} \\ -s_1 Q_{13}^* & -s_2 Q_{23}^* & 0 & Q_{23} & -Q_{13} \\ -s_1 s_2 Q_{14}^* & 0 & -s_2 Q_{23}^* & 0 & Q_{12} \\ 0 & -s_1 s_2 Q_{14}^* & s_1 Q_{13}^* & -s_1 s_2 Q_{12}^* & 0 \end{pmatrix}.$$

Hence we get the four-wave system

$$\begin{aligned} i(J_1 - J_2)Q_{12,t} - i(I_1 - I_2)Q_{12,x} - k s_2 Q_{13} Q_{23}^* &= 0 \\ iJ_1 Q_{13,t} - iI_1 Q_{13,x} - k(Q_{12} Q_{23} + s_2 Q_{14} Q_{23}^*) &= 0 \\ i(J_1 + J_2)Q_{14,t} - i(I_1 + I_2)Q_{14,x} - k Q_{13} Q_{23} &= 0 \\ iJ_2 Q_{23,t} - iI_2 Q_{23,x} - k s_1 (Q_{13}^* Q_{14} + s_2 Q_{12}^* Q_{13}) &= 0 \end{aligned}$$

where

$$k := J_1 I_2 - J_2 I_1.$$

In particular, if we choose $K_1 = \mathbb{1}$ we obtain that Q is an antihermitian matrix. The invariance condition (13) leads to the following form of the dressing factor

$$g = \mathbb{1} + \frac{A}{\lambda - \lambda^+} + \frac{K_1 S A^* (K_1 S)^{-1}}{\lambda - (\lambda^+)^*}$$

i.e., comparing with (9) we see that

$$B = K_1 S A^* (K_1 S)^{-1}, \quad \lambda^- = (\lambda^+)^*.$$

As a consequence of the linear dispersion law of the N -wave equation one can prove (see [8]) that it admits a \mathbb{Z}_2 reduction of the type $Q(x, t) = K_2 Q^T(x, t) K_2^{-1}$. This fact motivates us to pay special attention to this reduction as well.

Example 2. Another type of \mathbb{Z}_2 reduction is given by

$$\chi^+(x, t, \lambda) = K_2 \left\{ [\chi^-(x, t, -\lambda)]^T \right\}^{-1} K_2^{-1}.$$

Therefore we have the symmetry conditions

$$K_2 U(x, t, -\lambda)^T K_2^{-1} = -U(x, t, \lambda), \quad \Rightarrow Q = K_2 Q^T K_2^{-1}.$$

Consequently there are only four independent fields as shown below

$$Q = \begin{pmatrix} 0 & Q_{12} & Q_{13} & Q_{14} & 0 \\ s_1 s_2 Q_{12} & 0 & Q_{23} & 0 & Q_{14} \\ s_1 Q_{13} & s_2 Q_{23} & 0 & Q_{23} & -Q_{13} \\ s_1 s_2 Q_{14} & 0 & s_2 Q_{23} & 0 & Q_{12} \\ 0 & s_1 s_2 Q_{14} & -s_1 Q_{13} & s_1 s_2 Q_{12} & 0 \end{pmatrix}.$$

In particular if we choose $K = \mathbb{1}$ then Q is a symmetric matrix. The invariance condition implies that the dressing matrix gets the form

$$g(x, t, \lambda) = \mathbb{1} + \frac{A(x, t)}{\lambda - \lambda^+} - \frac{K_2 S A(x, t) (K_2 S)^{-1}}{\lambda + \lambda^+}.$$

Hence here the poles of the dressing factor form a doublet $\{\lambda^+, -\lambda^+\}$ whose residues are related by

$$B(x, t) = -K_2 S A(x, t) (K_2 S)^{-1}, \quad \lambda^- = -\lambda^+.$$

In this case the four-wave system reads

$$\begin{aligned} i(J_1 - J_2)Q_{12,t} - i(I_1 - I_2)Q_{12,x} + k s_2 Q_{13} Q_{23} &= 0 \\ iJ_1 Q_{13,t} - iI_1 Q_{13,x} + k Q_{23} (s_2 Q_{14} - Q_{12}) &= 0 \\ i(J_1 + J_2)Q_{14,t} - i(I_1 + I_2)Q_{14,x} - k Q_{13} Q_{23} &= 0 \\ iJ_2 Q_{23,t} - iI_2 Q_{23,x} + k s_1 Q_{13} (Q_{14} + s_2 Q_{12}) &= 0. \end{aligned}$$

In its turn the existence of such a reduction leads to the existence of a special class of solitons, the so-called breathers, in the case when there are two \mathbb{Z}_2 reductions (canonical one and another one of the type mentioned above) applied to the N -wave systems.

3. $\mathbb{Z}_2 \times \mathbb{Z}_2$ -Reductions and Breather-Type Solitons

3.1. Orthogonal Case

In this section we are going to demonstrate an algorithm how to obtain soliton solutions for a N -wave equation related to an orthogonal algebra (i.e., $\mathfrak{g} \equiv \mathfrak{so}(n, \mathbb{C})$) with $\mathbb{Z}_2 \times \mathbb{Z}_2$ reduction imposed on it. We shall follow the ideas presented by Zakharov and Mikhailov [15].

Let \mathfrak{g} be the orthogonal algebra $\mathfrak{so}(n, \mathbb{C})$. We remind the reader that the orthogonal algebra consists of all infinitesimal isometries in a complex space \mathbb{C}^n , i.e.,

$$\mathfrak{so}(n, \mathbb{C}) := \{C \in \mathfrak{gl}(n, \mathbb{C}); C^T S + S C = 0\}$$

where S is the metric in \mathbb{C}^n which determines a scalar product by the formula

$$(u, v) := u^T S v \quad u, v \in \mathbb{C}^n.$$

It is more convenient to work in a basis of \mathbb{C}^n such that the matrix of S in that basis reads

$$S = \sum_{k=1}^n (-1)^{k-1} (E_{k, 2n+1-k} + E_{2n+1-k, k}) \quad \text{for } \mathfrak{so}(2n).$$

and

$$S = \sum_{k=1}^n (-1)^{k-1} (E_{k,2n+1-k} + E_{2n+1-k,k}) + (-1)^n E_{nn} \quad \text{for } \mathfrak{so}(2n+1)$$

This choice of the basis ensures that corresponding Cartan subalgebras consist of diagonal matrices of the type

$$\begin{aligned} \mathfrak{h} &= \{J = \text{diag}(J_1, \dots, J_n, -J_n, \dots, -J_1)\} \quad \text{for } \mathfrak{so}(2n) \\ \mathfrak{h} &= \{J = \text{diag}(J_1, \dots, J_n, 0, -J_n, \dots, -J_1)\} \quad \text{for } \mathfrak{so}(2n+1). \end{aligned}$$

Let the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$ in the space of fundamental solutions of the linear problem is given by

$$\begin{aligned} \chi^-(x, t, \lambda) &= K_1 \left((\chi^+)^\dagger(x, t, \lambda^*) \right)^{-1} K_1^{-1} \\ \chi^-(x, t, \lambda) &= K_2 \left((\chi^+)^T(x, t, -\lambda) \right)^{-1} K_2^{-1} \end{aligned}$$

where $K_{1,2} \in \text{SO}(n, \mathbb{C})$ and $[K_1, K_2] = 0$. Consequently the potential $U(x, t, \lambda)$ satisfies the following symmetry conditions

$$K_1 U^\dagger(x, t, \lambda^*) K_1^{-1} = U(x, t, \lambda), \quad K_1 J^* K_1^{-1} = J \quad (14)$$

$$K_2 U^T(x, t, -\lambda) K_2^{-1} = -U(x, t, \lambda), \quad K_2 J K_2^{-1} = J. \quad (15)$$

In accordance with what we said in previous section the dressing factor g must be invariant under the action of $\mathbb{Z}_2 \times \mathbb{Z}_2$, i.e.,

$$K_1 \left(g^\dagger(x, t, \lambda^*) \right)^{-1} K_1^{-1} = g(x, t, \lambda) \quad (16)$$

$$K_2 \left(g^T(x, t, -\lambda) \right)^{-1} K_2^{-1} = g(x, t, \lambda). \quad (17)$$

To satisfy these requirements we choose a dressing factor as follows

$$\begin{aligned} g(x, t, \lambda) &= \mathbb{1} + \frac{A(x, t)}{\lambda - \lambda^+} + \frac{K_1 S A^*(x, t) (K_1 S)^{-1}}{\lambda - (\lambda^+)^*} \\ &\quad - \frac{K_2 S A(x, t) (K_2 S)^{-1}}{\lambda + \lambda^+} - \frac{K_1 K_2 A^*(x, t) (K_1 K_2)^{-1}}{\lambda + (\lambda^+)^*}. \end{aligned} \quad (18)$$

By taking the limit $\lambda \rightarrow \infty$ in equation (8) and taking into account the explicit formula (18) one can derive the following relation

$$[J, Q] = [J, Q_0 + A + K_1 S A^* S K_1 - K_2 S A S K_2 - K_1 K_2 A^* K_2 K_1]. \quad (19)$$

Thus, to find the soliton solution of the N -wave equation we have to know only one matrix-valued function – A . We will obtain A in two steps by deriving certain

algebraic and differential relations. First of all, let us recall that the dressing factor g belongs to the orthogonal group $SO(n, \mathbb{C})$, hence

$$g^{-1}(x, t, \lambda) = S^{-1}g^T(x, t, \lambda)S.$$

The equality $gg^{-1} = 1$ must hold identically with respect to λ and, therefore, A fulfills the algebraic restrictions

$$ASA^T = 0, \quad AS\omega^T + \omega SA^T = 0 \quad (20)$$

where

$$\omega := \mathbb{1} - \frac{K_2SASK_2}{2\lambda^+} + \frac{K_1SA^*SK_1}{2i\nu} - \frac{K_1K_2A^*K_2K_1}{2\mu}, \quad \lambda^+ = \mu + i\nu.$$

From (20) it follows that the matrix A is a degenerate one and it can be decomposed

$$A(x, t) = X(x, t)F^T(x, t)$$

where X and F are $n \times k$ ($1 \leq k < n$) matrices of maximal rank k . The equalities in (20) can be rewritten in terms of X and F as follows

$$F^T S F = 0, \quad X F^T S \omega^T + \omega S F X^T = 0$$

or introducing a $k \times k$ skew symmetric matrix $\alpha(x)$ the latter restriction reads

$$\left(\mathbb{1} - \frac{K_2SASK_2}{2\lambda^+} + \frac{K_1SA^*SK_1}{2i\nu} - \frac{K_1K_2A^*K_2K_1}{2\mu} \right) S F = X \alpha. \quad (21)$$

Another type of restrictions concerning the matrix-valued functions F and α comes from the λ -independence of the potential $[J, Q]$. If we express the potential in the equation (8) we get

$$[J, Q(x, t)] = -i\partial_x g g^{-1} + g[J, Q_0(x, t)]g^{-1} + \lambda(J - gJg^{-1}). \quad (22)$$

Annihilation of residues in (22) leads to the following differential equations

$$i\partial_x F^T(x, t) - F^T(x, t)([J, Q_0(x, t)] - \lambda^+ J) = 0$$

and

$$i\partial_x \alpha(x, t) + F^T(x, t)J S F(x, t) = 0.$$

After integration we obtain

$$F(x, t) = S\chi_0^+(x, t, \lambda^+)S F_0, \quad F_0 = \text{const} \\ \alpha(x, t) = F_0^T(\chi_0^+(x, t, \lambda^+))^{-1}\partial_\lambda \chi_0^+(x, t, \lambda^+)S F_0 + \alpha_0.$$

In the soliton case the fundamental solution is just a plane wave $e^{-i\lambda Jx}$. Therefore, the functions F and α get the form

$$F(x, t) = e^{i\lambda^+(Jx+It)}F_0, \quad \alpha(x, t) = iF_0^T(Jx + It)S F_0 + \alpha_0. \quad (23)$$

It remains to find the factor $X(x, t)$. For this purpose we apply simple algebraic manipulations on the linear equation (21) and as a result we obtain

$$\begin{aligned} SF &= X\alpha + Y\frac{(G, F)}{2\lambda^+} - Z\frac{(H, F)}{2i\nu} + W\frac{(N, F)}{2\mu} \\ SG &= X\frac{(F, G)}{2\lambda^+} + Y\alpha + Z\frac{(H, G)}{2\mu} - W\frac{(N, G)}{2i\nu} \\ SH &= X\frac{(F, H)}{2i\nu} + Y\frac{(G, H)}{2\mu} + Z\alpha^* + W\frac{(N, H)}{2(\lambda^+)^*} \\ SN &= X\frac{(F, N)}{2\mu} + Y\frac{(G, N)}{2i\nu} + Z\frac{(H, N)}{2(\lambda^+)^*} + W\alpha^* \end{aligned}$$

where we have introduced the auxiliary entities

$$\begin{aligned} Y &:= K_2SX, \quad Z := K_1SX^*, \quad W := K_1K_2X^* \\ G &:= K_2SF, \quad H := K_1SF^*, \quad N := K_1K_2F^*, \quad (F, H) := F^T SH. \end{aligned}$$

In matrix notations this system reads

$$(SF, SG, SH, SN) = (X, Y, Z, W) \begin{pmatrix} \alpha & a & b & c \\ a & \alpha & c & b \\ b^* & c^* & \alpha^* & a^* \\ c^* & b^* & a^* & \alpha^* \end{pmatrix}$$

where

$$a := \frac{(F, G)}{2\lambda^+}, \quad b := \frac{(F, H)}{2i\nu}, \quad c := \frac{(F, N)}{2\mu}.$$

To calculate X we just have to find the inverse matrix of the block matrix shown above. In the simplest case when $\text{rank}(X) = \text{rank}(F) = 1$ and $\alpha \equiv 0$ we have

$$\begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} 0 & a^* & b & -c \\ a^* & 0 & -c & b \\ -b & -c & 0 & a \\ -c & -b & a & 0 \end{pmatrix} \begin{pmatrix} SF \\ SG \\ SH \\ SN \end{pmatrix}$$

where

$$\Delta := |a|^2 + b^2 - c^2.$$

Finally, putting the result for X in (19) we obtain the breather solution

$$\begin{aligned} Q &= \frac{1}{\Delta} [(a^*K_2F + bK_1F^* - cK_1K_2SF^*)F^T \\ &\quad - K_2S(a^*K_2F + bK_1F^* - cK_1K_2SF^*)F^T SK_2 \\ &\quad + K_1S(aK_2F^* - bK_1F - cK_1K_2SF)F^+ SK_1 \\ &\quad - K_1K_2(aK_2F^* - bK_1F - cK_1K_2SF)F^+ K_2K_1]. \end{aligned}$$

If we apply this dressing procedure to the one-soliton solution we obtain a two-soliton solution. Iterating this process we can “generate” multisoliton solutions, i.e.,

$$0 \xrightarrow{g_{1s}} Q_{1s} \xrightarrow{g_{2s}} Q_{2s} \longrightarrow \cdots \xrightarrow{g_{ms}} Q_{ms}.$$

Example 3. Breather solution for a four-wave equation related to the $\mathfrak{so}(5)$ algebra.

Let us consider a N -wave equation associated with $\mathfrak{so}(5)$ algebra. Impose $\mathbb{Z}_2 \times \mathbb{Z}_2$ -type reductions as shown above with $K_1 = K_2 = \mathbb{1}$ and

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence we obtain a four-wave system

$$\begin{aligned} (J_1 - J_2)q_{1,t} - (I_1 - I_2)q_{1,x} + kq_2q_4 &= 0 \\ J_1q_{2,t} - I_1q_{2,x} + k(q_3 - q_1)q_4 &= 0 \\ (J_1 + J_2)q_{3,t} - (I_1 + I_2)q_{3,x} - kq_2q_4 &= 0 \\ J_2q_{4,t} - I_2q_{4,x} + k(q_1 + q_3)q_2 &= 0 \end{aligned}$$

where we have introduced more convenient real-valued fields as follows

$$Q_{12} = iq_1, \quad Q_{13} = iq_2, \quad Q_{14} = iq_3, \quad Q_{23} = iq_4$$

and

$$k := J_1I_2 - J_2I_1.$$

Its generic breather solution is

$$\begin{aligned} q_1 &= \frac{2}{\Delta} \Im[(a^* e^{i\lambda^+ K(1)} F_{0,1} + b e^{-i(\lambda^+)^* K(1)} F_{0,1}^* - c e^{i(\lambda^+)^* K(1)} F_{0,5}^*) \\ &\quad \times e^{i\lambda^+ K(2)} F_{0,2} + (a^* e^{-i\lambda^+ K(1)} F_{0,5} \\ &\quad + b e^{i(\lambda^+)^* K(1)} F_{0,5}^* - c e^{-i(\lambda^+)^* K(1)} F_{0,1}^*) e^{-i\lambda^+ K(2)} F_{0,4} \\ q_2 &= \frac{2}{\Delta} \Im[(a^* e^{i\lambda^+ K(1)} F_{0,1} + b e^{-i(\lambda^+)^* K(1)} F_{0,1}^* - c e^{i(\lambda^+)^* K(1)} F_{0,5}^*) F_{0,3} \\ &\quad - (a^* e^{-i\lambda^+ K(1)} F_{0,5} + b e^{i(\lambda^+)^* K(1)} F_{0,5}^* - c e^{-i(\lambda^+)^* K(1)} F_{0,1}^*) F_{0,3}] \\ q_3 &= \frac{2}{\Delta} \Im[(a^* e^{i\lambda^+ K(1)} F_{0,1} + b e^{-i(\lambda^+)^* K(1)} F_{0,1}^* - c e^{i(\lambda^+)^* K(1)} F_{0,5}^*) \\ &\quad \times e^{-i\lambda^+ K(2)} F_{0,4} + (a^* e^{-i\lambda^+ K(1)} F_{0,5} \\ &\quad + b e^{i(\lambda^+)^* K(1)} F_{0,5}^* - c e^{-i(\lambda^+)^* K(1)} F_{0,1}^*) e^{i\lambda^+ K(2)} F_{0,2}] \end{aligned}$$

$$q_4 = \frac{2}{\Delta} \Im [(a^* e^{i\lambda^+ K(2)} F_{0,2} + b e^{-i(\lambda^+)^* K(2)} F_{0,2}^* - c e^{i(\lambda^+)^* K(2)} F_{0,4}^*) F_{0,3} \\ + (a^* e^{-i\lambda^+ K(2)} F_{0,4} + b e^{i(\lambda^+)^* K(2)} F_{0,4}^* - c e^{-i(\lambda^+)^* K(2)} F_{0,2}^*) F_{0,3}]$$

where $K(1) = J_1 x + I_1 t$ and $K(2) = J_2 x + I_2 t$.

3.2. Symplectic Case

Consider a N -wave equation associated with a symplectic algebra $\mathfrak{sp}(2n, \mathbb{C})$. The soliton solutions for the canonical reduction were derived in [9].

Reminder. $\mathfrak{sp}(2n, \mathbb{C})$ is the Lie algebra of all infinitesimal symplectic morphisms in the complex symplectic vector space \mathbb{C}^{2n} , i.e.,

$$\mathfrak{sp}(2n, \mathbb{C}) := \{ \mathcal{X} \in \mathfrak{gl}(2n, \mathbb{C}); \mathcal{X}^T S + S \mathcal{X} = 0 \}$$

where S is a skew-symmetric bilinear form defined in \mathbb{C}^{2n} . We choose a basis in \mathbb{C}^{2n} such that S gets the form

$$S = \sum_{k=1}^n (-1)^{k+1} (E_{k, 2n+1-k} - E_{2n+1-k, k}).$$

The Cartan subalgebra of $\mathfrak{sp}(2n, \mathbb{C})$ consists of the diagonal matrices of the type

$$\mathfrak{h} = \{ B = \text{diag}(B_1, \dots, B_n, -B_n, \dots, -B_1) \}.$$

Consider the reduction group $\mathbb{Z}_2 \times \mathbb{Z}_2$ acting in $\text{Sp}(2n, \mathbb{C})$ as follows

$$\chi^-(x, t, \lambda) = K_1 \left((\chi^+)^{\dagger}(x, t, \lambda^*) \right)^{-1} K_1^{-1} \\ \chi^-(x, t, \lambda) = K_2 \left((\chi^+)^T(x, t, -\lambda) \right)^{-1} K_2^{-1}$$

where the symplectic matrices K_1 and K_2 are real and obey the restrictions

$$K_{1,2}^2 = \pm \mathbb{1}, \quad [K_1, K_2] = 0.$$

This leads to the symmetry conditions as shown in the previous subsection (see formulae (14) and (15)). In its turn the dressing factor $g(x, t, \lambda)$ is given by (18) and the soliton solution is

$$[J, Q] = [J, A - K_2 S A (K_2 S)^{-1} + K_1 S A^* (K_1 S)^{-1} - K_1 K_2 A^* (K_2 K_1)^{-1}].$$

One natural choice is $K_1 = K_2 = \mathbb{1}$ (we recall that $S = -S^T = -S^{-1}$). As a result we get

$$[J, Q(x, t)] = 2i[J, \Im(A(x, t) + SA(x, t)S)].$$

That is why we shall understand from now on this choice by default. Following the same procedures as in the orthogonal case we convince ourselves that the matrix-valued function A can be presented as a product of two $2n \times k$ -matrices as follows

$$A(x, t) = X(x, t)F^T(x, t).$$

The factor F is a solution of a simple linear differential equation leading to

$$F(x, t) = \left((\chi_0^+(x, t, \lambda^+))^T \right)^{-1} F_0.$$

In the soliton case $\chi_0^+(x, t) = e^{-i\lambda(Jx+It)}$ and therefore

$$F(x, t) = e^{i\lambda^+(Jx+It)} F_0.$$

To obtain the factor X we solve a linear system which in the simplest case when $\text{rank}(X) = \text{rank}(F) = 1$ reads

$$\begin{pmatrix} SF \\ SG \\ SH \\ SN \end{pmatrix} = \begin{pmatrix} \alpha & a & -b & -c \\ -a & \alpha & c & -b \\ -b & c & \alpha^* & a^* \\ -c & -b & -a^* & \alpha^* \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \\ W \end{pmatrix}$$

where

$$Y := SX, \quad Z := SX^*, \quad W := -X^*, \quad G := SF, \quad H := SF^*, \quad N := -F^*$$

$$\alpha := iF_0^T(Jx + It)SF_0, \quad a := \frac{F^T F}{2\lambda^+}, \quad b := \frac{F^\dagger F}{2i\nu}, \quad c := \frac{F^\dagger SF}{2\mu}.$$

Hence we obtain that

$$X = \frac{1}{\Delta} \left(\tilde{\alpha}SF - \tilde{a}SG - \tilde{b}SH - \tilde{c}SN \right)$$

where

$$\begin{aligned} \tilde{\alpha} &:= \alpha^*(|\alpha|^2 - b^2 - c^2) + \alpha(a^*)^2 \\ \tilde{a} &:= a^*(|a|^2 + b^2 + c^2) + a(\alpha^*)^2 \\ \tilde{b} &:= b(|a|^2 - |\alpha|^2 + b^2 + c^2) - 2c\Re(a\alpha^*) \\ \tilde{c} &:= c(|a|^2 - |\alpha|^2 + b^2 + c^2) + 2b\Re(a\alpha^*) \\ \Delta &:= |a|^4 + 2\Re(a^2(\alpha^*)^2) + |\alpha|^4 + 2(|a|^2 - |\alpha|^2 + b^2 + c^2)(b^2 + c^2). \end{aligned}$$

The soliton solution is given by

$$Q = \frac{2i}{\Delta} \Im \left[\left(\tilde{\alpha}SF + \tilde{a}F + \tilde{b}F^* + \tilde{c}SF^* \right) F^T + S \left(\tilde{\alpha}SF + \tilde{a}F + \tilde{b}F^* + \tilde{c}SF^* \right) F^T S \right] \quad (24)$$

This in principle solves the given task.

4. Conclusion

We derived $\mathbb{Z}_2 \times \mathbb{Z}_2$ -reduced N -wave systems that admit a special type of soliton solutions – a breather type solutions. In order to calculate them we have modified the classical Zakharov–Shabat dressing method. Here we have illustrated our result by the breather solution of the four-wave equation associated with the orthogonal $\mathfrak{so}(5)$ algebra. The method can be applied also for the symplectic algebras.

To each of these solutions there corresponds a quadruplet $\{\pm\lambda^+, \pm(\lambda^+)^*\}$ of symmetrically situated discrete eigenvalues of the operator L . These eigenvalues are determined by the poles of the resolvent and dressing matrices.

One may expect that along with the breather solutions, our $\mathbb{Z}_2 \times \mathbb{Z}_2$ -reduced N -wave systems would allow also for “doublet” solitons for which λ^+ is purely imaginary. This idea along with the analysis of the different types of soliton solutions [6] will be presented elsewhere.

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