Eighth International Conference on
Geometry, Integrability and Quantization
June 9–14, 2006, Varna, Bulgaria
Ivaïlo M. Mladenov and Manuel de León, Editors
SOFTEX, Sofia 2007, pp 312–321

ON THE TRANSLATIONALLY-INVARIANT SOLUTIONS OF THE MEMBRANE SHAPE EQUATION

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Abstract. The membrane shape equation derived by Helfrich and Ou-Yang describes the equilibrium shapes of biomembranes, built by bilayers of amphiphilic molecules, in terms of the mean and Gaussian curvatures of their middle-surfaces. Here, we present a new class of translationally-invariant solutions to this equation in terms of the elliptic functions which completes the solutions found earlier. In this way, all translationally-invariant solutions to the membrane shape equation are determined. Special attention is paid to those translationally-invariant solutions of the membrane shape equation which determine closed cylindrical (tube-like) surfaces (membrane shapes). Several examples of such surfaces are presented.

1. Introduction

Within the framework of the Helfrich spontaneous curvature model [3], the equilibrium shapes of a biomembrane, assumed as a bilayer of amphiphilic molecules (phospholipids, for instance), are described in terms of the mean H and Gaussian K curvatures of its middle-surface $\mathcal{S}$ by the **membrane shape equation** [7, 8]

$$2k_c\Delta H + k_c\,(2H + \mathbb{h})\,(2H^2 - \mathbb{h}H - 2K) - 2\lambda H + p = 0 \qquad (1)$$

where k_c, $\mathbb{h}$ and λ are real constants representing the bending rigidity, spontaneous curvature and tensile stress of the membrane, respectively, while p is the osmotic pressure difference between the outer and inner media assumed to be a real constant too. Here, Δ is the **Laplace–Beltrami operator** on the surface $\mathcal{S}$.

In a previous study by the present authors (see [10]), it is established that the six-parameter group of motions in the three dimensional Euclidean space is the largest group of point transformations admitted by the membrane shape equation in Mongé representation. In that work, all types of non-equivalent group-invariant solutions of this equation are identified via an optimal system of one-dimensional subalgebras of the symmetry algebra and the corresponding reduced equations are derived. In [10], special attention is paid to the **translationally-invariant solutions** of the membrane shape equation assuming that the osmotic pressure difference $p \neq 0$ since the case $p = 0$ is thoroughly studied elsewhere (see [4, 5, 9, 11]). All translationally-invariant solutions to the membrane shape equation that can be expressed in elementary functions and some solutions that are given in terms of elliptic functions are obtained in [10]. The aim of this study is to determine all other translationally-invariant solutions of the membrane shape equation in Mongé representation.

2. Translationally-Invariant Solutions

In [10], it is shown that the translationally-invariant solutions of the membrane shape equation (1) in Mongé representation correspond to cylindrical surfaces in $\mathbb{R}^3$ whose directrices are plane curves Γ of curvature $\mathbb{k}(s) = 2H(s)$ that satisfies the equation

$$2\frac{\mathrm{d}^2\mathbb{k}}{\mathrm{d}s^2} + \mathbb{k}^3 - \mu\mathbb{k} - \sigma = 0 \tag{2}$$

where

$$\mu = \mathbb{h}^2 + \frac{2\lambda}{k_c}, \qquad \sigma = -\frac{2p}{k_c}$$

s being the arc length of the respective curve Γ. The generatrices of the foregoing cylindrical surfaces are perpendicular to the plane the directrices Γ lie in. Once a solution $\mathbb{k}(s)$ of equation (2) is known in an explicit form, it is possible to recover the embedding $s \longmapsto (x(s), z(s))$ of the corresponding curve Γ in the XOZ plane (up to a rigid motion) by solving the system

$$\begin{aligned} \frac{\mathrm{d}x(s)}{\mathrm{d}s}\frac{\mathrm{d}^2z(s)}{\mathrm{d}s^2} - \frac{\mathrm{d}^2x(s)}{\mathrm{d}s^2}\frac{\mathrm{d}z(s)}{\mathrm{d}s} &= \mathbb{k}(s) \\ \left(\frac{\mathrm{d}x(s)}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}z(s)}{\mathrm{d}s}\right)^2 &= 1. \end{aligned} \tag{3}$$

Thus, the main problem to solve is to find the solutions of equation (2).

This equation is studied by Arreaga *et al* [1] with the aim to determine the equilibria of an elastic loop in the plane subject to the constraints of fixed length and fixed enclosed area. In the three dimensional case considered here, each such loop will determine a directrix Γ of a cylindrical surface that corresponds to a translationally-invariant solution of the membrane shape equation. In [1], the determination of the curvature $\Bbbk$ at equilibrium is reduced to the study of the motion of a particle in a quartic potential. Indeed, equation (2) is the Euler–Lagrange equation associated with the functional

$$F(\Bbbk) = \int (T - U)\,\mathrm{d}s, \qquad T = \frac{1}{2}\left(\frac{\mathrm{d}\Bbbk}{\mathrm{d}s}\right)^2, \qquad U = \frac{1}{8}\Bbbk^4 - \frac{1}{4}\mu\Bbbk^2 - \frac{1}{2}\sigma\Bbbk$$

in which $\Bbbk$, T and U can be thought of as the displacement, kinetic energy and potential energy, respectively, of some fictitious particle. In this setting, s plays the role of the time. Using this analogy, the authors succeeded in obtaining a purely geometric construction for determination of the curvature of the loop passing through a given point of the plane without a reference to explicit expressions for the solutions of equation (2).

However, in our opinion, the knowledge of the solutions of this equation in an explicit form is an important and powerful tool in determination of the surfaces that are translationally-invariant solutions of the membrane shape equation. For this reason, the authors determined some explicit solutions in the previous study [10] and complete this problem here.

Evidently, equation (2) admits the one-parameter group of translations of the independent variable s as a variational symmetry group. Hence, by virtue of Noether's theorem, there is a conservation law of density $E = T + U$, further referred to as the total energy, that is a first integral

$$\frac{\mathrm{d}E}{\mathrm{d}s} = 0 \tag{4}$$

that holds on its smooth solutions. In characteristic form (see [6]), the above conservation law reads

$$\frac{\mathrm{d}E}{\mathrm{d}s} = \frac{1}{2}\frac{\mathrm{d}\Bbbk}{\mathrm{d}s}\left(2\frac{\mathrm{d}^2\Bbbk}{\mathrm{d}s^2} + \Bbbk^3 - \mu\Bbbk - \sigma\right).$$

Therefore, each solution $\Bbbk = \Bbbk(s)$ of equation (2), which is not identically a constant, corresponds to a certain real value of the total energy E, and satisfies the equation

$$\left(\frac{\mathrm{d}\Bbbk}{\mathrm{d}s}\right)^2 - P(\Bbbk) = 0, \qquad P(\Bbbk) = 2E - \frac{1}{4}\Bbbk^4 + \frac{1}{2}\mu\Bbbk^2 + \sigma\Bbbk. \tag{5}$$

Bearing in mind that the coefficients of the polynomial $P(\Bbbk)$ are real numbers, the foregoing value of the total energy E is such that the polynomial $P(\Bbbk)$ has at least two different real roots, otherwise the function $P(\Bbbk(s))$ could not take non-negative values as required by the first relation in (5). In the light of the above, there are only two possibilities for the roots of the polynomial $P(\Bbbk)$, namely: I) the four roots are real; **II**) two of the roots are real and the other two are a complex conjugate pair. Now, we denote by α, β, γ and δ the roots of the polynomial $P(\Bbbk)$ and specialize to the cases in which $P(\Bbbk)$ has simple roots, so that without loss of generality in the first case we consider $\alpha < \beta < \gamma < \delta$ and in the second one $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ and $\gamma, \delta \in \mathbb{C} \setminus \mathbb{R}$, $\delta = \bar{\gamma}$. Thus, the polynomial $P(\Bbbk)$ is nonnegative in the intervals $\alpha \leq \Bbbk \leq \beta$ and $\gamma \leq \Bbbk \leq \delta$, in case **I**), and in the interval $\alpha \leq \Bbbk \leq \beta$, in case **II**). These situations are depicted as c) and d) cases in Fig. 1.

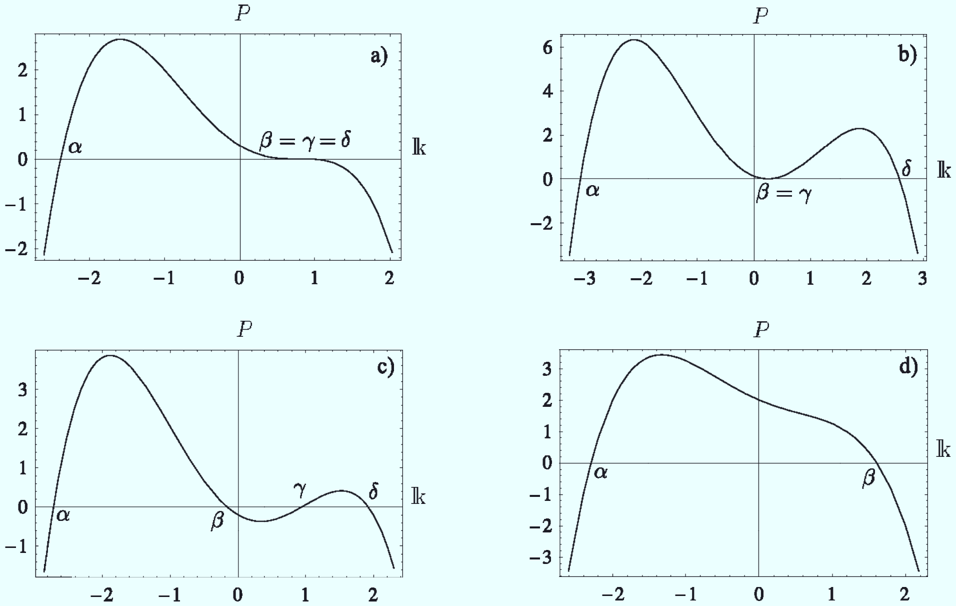

Figure 1. Four different types of polynomials $P(\Bbbk)$: a) two real roots – one simple and one triple, $\mu = 1.89$, $\sigma = -1$, $E = 2.72$; b) three real roots – two simple and one double $\mu = 4.06$, $\sigma = -1$, $E = 0.06$; c) four simple real roots, $\mu = 3$, $\sigma = -1$, $E = -0.1$; d) two simple real roots and a pair of complex conjugated roots, $\mu = 1$, $\sigma = -1$, $E = 1$.

All solutions in elementary functions of equation (2) corresponding to the multiple roots of the polynomial $P(\Bbbk)$ are presented in [10]. That is why, only the cases, in which $P(\Bbbk)$ has simple roots are considered here.

It should be noted that the roots α, β, γ and δ of the polynomial $P(\mathbf{k})$ can be expressed through its coefficients μ, σ and E as follows

$$\alpha = \sqrt{\frac{\Omega}{2}} + \sqrt{\mu + \sigma\sqrt{\frac{2}{\Omega}} - \frac{\Omega}{2}}, \qquad \beta = \sqrt{\frac{\Omega}{2}} - \sqrt{\mu + \sigma\sqrt{\frac{2}{\Omega}} - \frac{\Omega}{2}}$$

$$\gamma = -\sqrt{\frac{\Omega}{2}} + \sqrt{\mu - \sigma\sqrt{\frac{2}{\Omega}} - \frac{\Omega}{2}}, \qquad \delta = -\sqrt{\frac{\Omega}{2}} - \sqrt{\mu - \sigma\sqrt{\frac{2}{\Omega}} - \frac{\Omega}{2}}$$

where

$$\Omega = \frac{(\mu + \sqrt[3]{\kappa})^2 - 2^3 3E}{3\sqrt[3]{\kappa}}$$
$$\kappa = 3\left(3^2\sigma^2 + \sqrt{\chi}\right) - \mu\left(\mu^2 + 2^3 3^2 E\right)$$
$$\chi = 3\left(2^3 E\left(\left(\mu^2 + 8E\right)^2 - 3^2 2\mu\sigma^2\right) - \sigma^2\left(2\mu^3 - 3^3\sigma^2\right)\right).$$

By Vieta's formulas we also have

$$\begin{aligned} \mu &= -\frac{1}{2}(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta) \\ \sigma &= \frac{1}{4}(\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta) \\ E &= -\frac{1}{8}\alpha\beta\gamma\delta. \end{aligned} \tag{6}$$

and

$$\alpha + \beta + \gamma + \delta = 0 \tag{7}$$

due to the absence of a term with $\Bbbk^3$ in the polynomial $P(\Bbbk)$. The condition $\sigma \neq 0$ implies

$$\alpha + \beta \neq 0, \qquad \alpha + \gamma \neq 0, \qquad \beta + \gamma \neq 0. \tag{8}$$

3. New Explicit Solutions

Explicit expressions for the solutions of equation (2) are given in Lemma 1 and Lemma 2 for cases I) and II), respectively. Lemma 3 shows that any other periodic solution to equation (2) coincides (up to a shift of the independent variable) with one of these solutions.

Lemma 1. *Given μ and σ, let E be such that the roots α, β, γ and δ of the corresponding polynomial $P(\Bbbk)$ are real numbers ordered in the following manner: $\alpha < \beta < \gamma < \delta$. Consider the functions*

$$\Bbbk_1(s) = \delta - \frac{(\delta - \alpha)(\delta - \beta)}{(\delta - \beta) + (\beta - \alpha)\mathrm{sn}^2(us, k)} \tag{9}$$

$$\Bbbk_2(s) = \beta + \frac{(\gamma - \beta)(\delta - \beta)}{(\delta - \beta) - (\delta - \gamma)\mathrm{sn}^2(us, k)} \tag{10}$$

of the real variable s in which the parameter u and the ***elliptic module*** *k are given by the formulas*

$$u = \frac{1}{4}\sqrt{(\gamma-\alpha)(\delta-\beta)}, \qquad k = \sqrt{\frac{(\beta-\alpha)(\delta-\gamma)}{(\gamma-\alpha)(\delta-\beta)}}. \tag{11}$$

Then, both functions (9) *and* (10) *are real-valued, they are periodic with period* $(2/u)K(k)$ *and satisfy equation* (2).

Proof: It is easy to see that the condition $\alpha < \beta < \gamma < \delta \in \mathbb{R}$ and expressions (11) imply $u \in \mathbb{R}$ and $0 < k < 1$. Therefore, both functions (9) and (10) are real-valued. Evidently, these functions are periodic due to the fact that the function $\mathrm{sn}^2(us, k)$ is periodic with period $(2/u)K(k)$. Finally, substituting each of the above functions into equation (2), one can easily verify that they satisfy it. □

Lemma 2. *Given μ and σ, let E be such that two of the roots, say α and β, of the corresponding polynomial $P(\mathbb{k})$ are real numbers ordered as follows: $\alpha < \beta$, while the other two roots, denoted by γ and δ, are a complex conjugate pair, that is $\delta = \bar{\gamma}$. Consider the function*

$$\mathbb{k}_3(s) = \frac{(A\beta + B\alpha) + (A\beta - B\alpha)\mathrm{cn}(\upsilon s, \kappa)}{(A+B) + (A-B)\mathrm{cn}(\upsilon s, \kappa)} \tag{12}$$

of the real variable s in which the parameters A, B, υ and the elliptic module κ are given by the formulas

$$A = \sqrt{4\eta^2 + (3\alpha+\beta)^2}, \qquad B = \sqrt{4\eta^2 + (\alpha+3\beta)^2} \tag{13}$$

$$\upsilon = \frac{1}{4}\sqrt{AB}, \qquad \kappa = \sqrt{\frac{1}{2} - \frac{4\eta^2 + (3\alpha+\beta)(\alpha+3\beta)}{2AB}} \tag{14}$$

where $\eta = (\gamma - \bar{\gamma})/2\mathrm{i}$. Then, this function takes real values, it is periodic with period $(4/\upsilon)K(\kappa)$ and satisfies equation (2).

Proof: Evidently, in this case, the condition $\alpha < \beta \in \mathbb{R}$ and expressions (13) and (14) imply $\upsilon \in \mathbb{R}$ and $0 < \kappa < 1$. Hence, function (12) is real-valued. Obviously, this function is periodic because the Jacobian elliptic function $\mathrm{cn}(\upsilon s, \kappa)$ is periodic with period $(4/\upsilon)K(\kappa)$. Finally, substituting the above function into equation (2), one can easily verify that the latter equation is satisfied. □

Lemma 3. *Given μ, σ and E, let $\tilde{\mathbb{k}}_i(s)$, $i = 1, 2$, be two periodic real-valued functions of the real variable s with periods T_i, respectively. Let*

$$\frac{\mathrm{d}\tilde{\mathbb{k}}_i(s)}{\mathrm{d}s} = \sqrt{P(\tilde{\mathbb{k}}_i(s))}, \qquad s \in I_i \equiv \left[0, \frac{T_i}{2}\right]$$

and $a = \tilde{\Bbbk}_1(0) = \tilde{\Bbbk}_2(0)$ *is the minimum value of both functions. Then*

$$\tilde{\Bbbk}_1(s) = \tilde{\Bbbk}_2(s).$$

Proof: Obviously the functions $\tilde{\Bbbk}_i(s)$ are invertible for $s \in I_i$, respectively. Let us denote the corresponding inverse functions by $\tilde{\Bbbk}_i^{-1}(\tilde{\Bbbk})$. Then, differentiating the relations

$$s = \tilde{\Bbbk}_i^{-1}(\tilde{\Bbbk}) = \tilde{\Bbbk}_i^{-1}(\tilde{\Bbbk}_i(s))$$

one can see that for each $\tilde{\Bbbk} \in (a, b)$ where $b = \min\left(\tilde{\Bbbk}_1\left(\frac{T_1}{2}\right), \tilde{\Bbbk}_2\left(\frac{T_2}{2}\right)\right)$

$$\frac{\mathrm{d}\tilde{\Bbbk}_i^{-1}(\tilde{\Bbbk})}{\mathrm{d}\tilde{\Bbbk}} = \frac{1}{\sqrt{P(\tilde{\Bbbk})}}.$$

Thus, for $\tilde{\Bbbk} \in (a, b)$

$$\frac{\mathrm{d}}{\mathrm{d}\tilde{\Bbbk}}\left(\tilde{\Bbbk}_1^{-1}(\tilde{\Bbbk}) - \tilde{\Bbbk}_2^{-1}(\tilde{\Bbbk})\right) = 0$$

and therefore there exists a real constant s_0 such that

$$\tilde{\Bbbk}_2^{-1}(\tilde{\Bbbk}) = \tilde{\Bbbk}_1^{-1}(\tilde{\Bbbk}) + s_0.$$

So,

$$\tilde{\Bbbk}_2^{-1}\left(\tilde{\Bbbk}_1(s)\right) = \tilde{\Bbbk}_1^{-1}\left(\tilde{\Bbbk}_1(s)\right) + s_0 = s + s_0, \qquad s \in \min(I_1, I_2)$$

and hence

$$\tilde{\Bbbk}_1(s) = \tilde{\Bbbk}_2(s + s_0), \qquad s \in \min(I_1, I_2).$$

For $s = 0$ this implies $\tilde{\Bbbk}_2(s_0) = \tilde{\Bbbk}_1(0) = \tilde{\Bbbk}_2(0)$ which means that s_0 is a period of the function $\tilde{\Bbbk}_2(s)$ and hence the above relation reads

$$\tilde{\Bbbk}_1(s) = \tilde{\Bbbk}_2(s), \qquad s \in \mathbb{R}$$

which completes the proof. □

4. Curves and Membrane Shapes

Now, having obtained the solutions of equation (2) in explicit form, one can proceed with constructing the corresponding curves Γ in the XOZ plane by solving system (3). Thus, given μ, σ and a solution $\Bbbk = \Bbbk(s)$ of the corresponding equation (2), without loss of generality, one can represent system (3) in the form

$$\varphi(s) = \int \Bbbk(s)\,\mathrm{d}s, \qquad \frac{\mathrm{d}x(s)}{\mathrm{d}s} = \cos\varphi(s), \qquad \frac{\mathrm{d}z(s)}{\mathrm{d}s} = \sin\varphi(s). \tag{15}$$

Then, using the results presented in [1, 2], which can be cast in the form

$$x(s)\frac{\mathrm{d}x(s)}{\mathrm{d}s} + z(s)\frac{\mathrm{d}z(s)}{\mathrm{d}s} = \frac{2}{\sigma}\frac{\mathrm{d}\Bbbk(s)}{\mathrm{d}s}$$

$$x(s)\frac{\mathrm{d}z(s)}{\mathrm{d}s} - z(s)\frac{\mathrm{d}x(s)}{\mathrm{d}s} = \frac{1}{\sigma}(\Bbbk(s)^2 - \mu)$$

and taking into account relations (15), to write down the explicit expressions

$$\begin{aligned} x(s) &= \frac{2}{\sigma}\frac{\mathrm{d}\Bbbk(s)}{\mathrm{d}s}\cos\varphi(s) + \frac{1}{\sigma}\left(\Bbbk(s)^2 - \mu\right)\sin\varphi(s) \\ z(s) &= \frac{2}{\sigma}\frac{\mathrm{d}\Bbbk(s)}{\mathrm{d}s}\sin\varphi(s) - \frac{1}{\sigma}\left(\Bbbk(s)^2 - \mu\right)\cos\varphi(s) \end{aligned} \tag{16}$$

for the components of the position vector of the corresponding curve Γ. Computing the respective integrals in (15) one obtains

$$\varphi_1(s) = \delta s - \frac{\delta - \alpha}{u}\Pi\left(\frac{\beta - \alpha}{\beta - \delta}, \mathrm{am}(us, k), k\right) \tag{17}$$

$$\varphi_2(s) = \beta s - \frac{\beta - \gamma}{u}\Pi\left(\frac{\delta - \gamma}{\delta - \beta}, \mathrm{am}(us, k), k\right) \tag{18}$$

$$\begin{aligned} \varphi_3(s) = {} & \frac{A\beta - B\alpha}{A - B}s + \frac{(A+B)(\alpha - \beta)}{2v(A-B)}\Pi\left(-\frac{(A-B)^2}{4AB}, \mathrm{am}(vs, \kappa), \kappa\right) \\ & - \frac{\alpha - \beta}{2v\sqrt{\kappa^2 + \frac{(A-B)^2}{4AB}}}\arctan\left(\sqrt{\kappa^2 + \frac{(A-B)^2}{4AB}}\,\frac{\mathrm{sn}(vs, \kappa)}{\mathrm{dn}(vs, \kappa)}\right) \end{aligned} \tag{19}$$

for the solutions $\Bbbk_i(s)$, $i = 1, 2, 3$, specified in equations (9), (10) and (12). Thus, given any solution in one of the above forms, we can draw the corresponding curve by substituting the respective angle (17), (18) or (19) in the expressions for the components of the position vector (16). Closed curves are generated by the respective solution presented in (9), (10) or (12) if there exist some integers m and n such that

$$\begin{aligned} \frac{\delta}{u}K(k) + \frac{\alpha - \delta}{u}\Pi\left(\frac{\alpha - \beta}{\delta - \beta}, k\right) &= \frac{\pi m}{n} \\ \frac{\beta}{u}K(k) + \frac{\gamma - \beta}{u}\Pi\left(\frac{\gamma - \delta}{\beta - \delta}, k\right) &= \frac{\pi m}{n} \\ \frac{2(A\beta - B\alpha)}{v(A-B)}K(\kappa) + \frac{(A+B)(\alpha - \beta)}{v(A-B)}\Pi\left(-\frac{(A-B)^2}{4AB}, \kappa\right) &= \frac{\pi m}{n}. \end{aligned}$$

Several examples of closed curves Γ corresponding to solutions of form (9) and (10) of equation (2) with coefficients $\mu = 11.82$ and $\sigma = -13.3$ are presented in Fig. 2 and Fig. 3. It is worthy to underline that in this case, two distinct curves correspond to the same value of the total energy E.

Other examples of closed curves Γ corresponding to solutions of form (12) of equation (2) with coefficients $\mu = -1$ and $\sigma = 1/2$ are presented in Fig. 4.

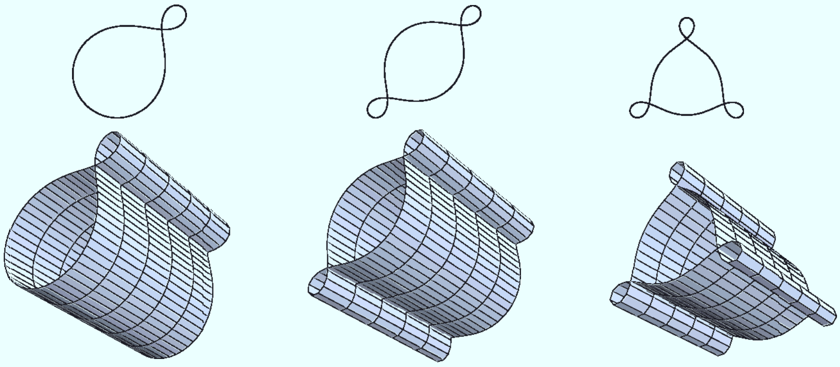

Figure 2. Closed self-intersecting curves and cylindrical surfaces obtained by the solution (9) corresponding to $\mu = 11.82$, $\sigma = -13.3$.

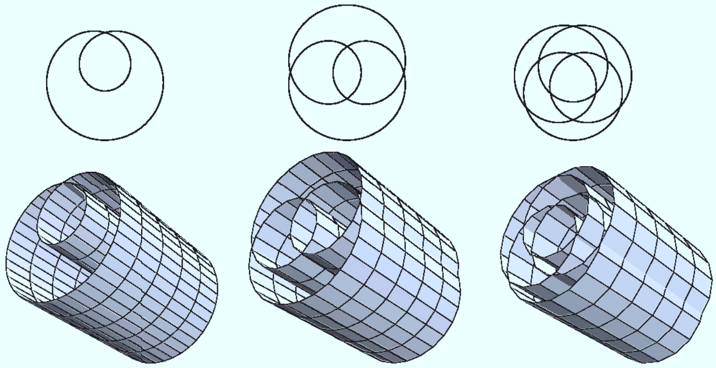

Figure 3. Closed self-intersecting curves and cylindrical surfaces obtained by the solution (10) corresponding to $\mu = 11.82$, $\sigma = -13.3$.

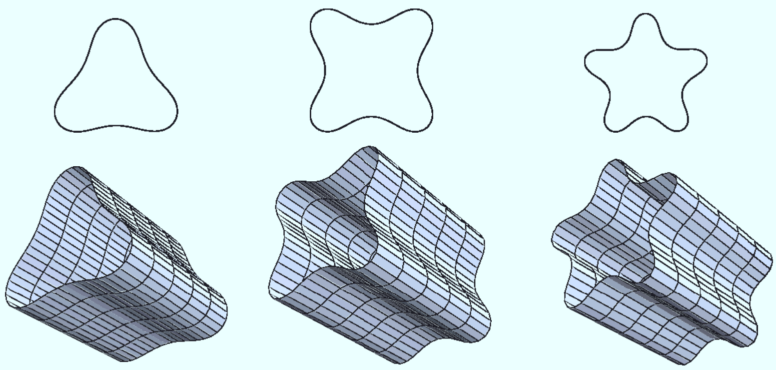

Figure 4. Closed non-self-intersecting curves and cylindrical surfaces obtained by the solution (12) corresponding to $\mu = -1$, $\sigma = 1/2$.

Acknowledgements

This research is partially supported by the Bulgarian National Science Foundation under a grant B–1531/2005 and the contract # 23/2006 between Bulgarian and Polish Academies of Sciences. The authors are grateful to Mrs. Mariana Hadzilazova for her help with graphics files.

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