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A property of Petrov's diffusion*

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Abstract

Petrov constructed a diffusion process in the compact Kingman simplex whose unique stationary distribution is the two-parameter Poisson–Dirichlet distribution of Pitman and Yor. We show that the subset of the simplex comprising vectors whose coordinates sum to 1 is the natural state space for the process. In fact, the complementary set acts like an entrance boundary.

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1 Introduction

Petrov (2009) constructed an infinite-dimensional diffusion process in the compact Kingman simplex

$$\overline{\nabla}_\infty := \left\{ x = (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0,\ \sum_{i=1}^\infty x_i \leq 1 \right\}$$

depending on two parameters, α and θ with $0 \leq \alpha < 1$ and $\theta > -\alpha$. Its generator is[1]

$$A := \frac{1}{2} \sum_{i,j=1}^\infty x_i(\delta_{ij} - x_j) \frac{\partial^2}{\partial x_i \, \partial x_j} - \frac{1}{2} \sum_{i=1}^\infty (\theta x_i + \alpha) \frac{\partial}{\partial x_i}$$

acting on the subalgebra of $C(\overline{\nabla}_\infty)$ generated by the sequence of functions $\varphi_1, \varphi_2, \varphi_3, \ldots$ defined by

$$\varphi_m(x) := \sum_{i=1}^\infty x_i^m, \quad m = 2, 3, \ldots, \qquad \varphi_1(x) := 1.$$

More precisely, for $\varphi \in \mathscr{D}(A)$, $A\varphi$ is evaluated on the dense subset

$$\nabla_\infty := \left\{ x = (x_1, x_2, \ldots) : x_1 \geq x_2 \geq \cdots \geq 0,\ \sum_{i=1}^\infty x_i = 1 \right\}$$

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[1] Petrov omitted the common factor of $\frac{1}{2}$ for simplicity but we include it so that formulas are consistent with those in the literature.

and extended to $\overline{\nabla}_\infty$ by continuity. For example, $A\varphi_2 = 1 - \alpha - (1+\theta)\varphi_2$. The unique stationary distribution of Petrov's diffusion is Pitman and Yor's (1997) two-parameter generalization of the Poisson–Dirichlet distribution, which we denote by $\mathrm{PD}_{\alpha,\theta}$ and regard as a Borel probability measure on $\overline{\nabla}_\infty$ that is concentrated on ∇_∞.

The special case $\alpha = 0$ (and hence $\theta > 0$) is the unlabeled infinitely-many-neutral-alleles diffusion model of population genetics; see Ethier and Kurtz (1981). Its unique stationary distribution is of course the (one-parameter) Poisson–Dirichlet distribution, $\mathrm{PD}_{0,\theta}$, of Kingman (1975).

Feng, Sun, Wang, and Xu (2011) derived an explicit formula for the transition density $p(t,x,y)$ of Petrov's diffusion with respect to $\mathrm{PD}_{\alpha,\theta}$, which had earlier been done in the special case $\alpha = 0$ by Ethier (1992). Recently, Zhou (2013) found an elegant simplification of this formula. However, all we will need here is the fact that the transition function $P(t,x,dy)$ is, for each $x \in \overline{\nabla}_\infty$ and $t > 0$, absolutely continuous with respect to $\mathrm{PD}_{\alpha,\theta}$:

$$P(t,x,\cdot) \ll \mathrm{PD}_{\alpha,\theta}, \quad x \in \overline{\nabla}_\infty,\ t > 0. \tag{1.1}$$

In particular, letting $\{X_t,\ t \geq 0\}$ denote Petrov's diffusion, it follows that $P_x(X_t \in \nabla_\infty) = 1$ for every $x \in \overline{\nabla}_\infty$ and $t > 0$, where the subscript x denotes the initial state. A question left open by Petrov (2009) is whether the stronger statement,

$$P_x(X_t \in \nabla_\infty \text{ for all } t > 0) = 1, \quad x \in \overline{\nabla}_\infty, \tag{1.2}$$

holds. This would tell us, in particular, that ∇_∞ is the natural state space for the process.

In the special case $\alpha = 0$, (1.2) was proved by Ethier and Kurtz (1981). It was later realized that this result has a simple interpretation. The unlabeled infinitely-many-neutral-alleles diffusion model has a more informative labeled version, namely the Fleming–Viot process in $\mathscr{P}(S)$ (the set of Borel probability measures on the compact metric space S with the topology of weak convergence) with mutation operator

$$Bg(z) := \frac{1}{2}\theta \int_S (g(\zeta) - g(z))\, \nu_0(d\zeta),$$

where $\nu_0 \in \mathscr{P}(S)$ is nonatomic. The unlabeled model is a transformation of the labeled one. The transformation takes $\mu \in \mathscr{P}(S)$ to $x \in \overline{\nabla}_\infty$, where x is the vector of descending order statistics of the sizes of the atoms of μ. Then (1.2) is equivalent to the assertion that the Fleming–Viot process, regardless of its initial state, instantly becomes purely atomic and remains so forever.

If $\alpha > 0$, there is no such interpretation of (1.2) because whether there is a Fleming–Viot process corresponding to Petrov's diffusion is unknown. In fact, this is an open problem that was posed by Feng (2010, p. 112) (see also Feng and Sun 2010).

Notice that another way to express (1.2) is to say that $\overline{\nabla}_\infty - \nabla_\infty$ acts like an entrance boundary for the diffusion. Technically, this is not quite accurate because ∇_∞ has no interior, so its boundary is all of $\overline{\nabla}_\infty$. But what we mean is simply that, starting at a state $x \in \overline{\nabla}_\infty - \nabla_\infty$, the process instantly enters ∇_∞ and never exits.

A weaker version of (1.2) was obtained by Feng and Sun (2010) using the theory of Dirichlet forms. They showed that

$$\int_{\overline{\nabla}_\infty} P_y(X_t \in \nabla_\infty \text{ for all } t \geq 0)\, \mathrm{PD}_{\alpha,\theta}(dy) = 1, \tag{1.3}$$

which is to say that the stationary version of Petrov's diffusion has ∇_∞ as its natural state space. Their argument is similar to that of Schmuland (1991), who treated the special case $\alpha = 0$ and gave a more detailed proof. Of course an equivalent way to state (1.3) is

$$P_y(X_t \in \nabla_\infty \text{ for all } t \geq 0) = 1 \text{ a.e.-PD}_{\alpha,\theta}(dy). \tag{1.4}$$

In the next section we will see that (1.2) follows easily from (1.1), (1.4), and the (time-homogeneous) Markov property.

2 Entrance boundary property

We are now ready for the proof of (1.2).

Theorem 2.1. *Eq.* (1.2) *holds for Petrov's diffusion.*

Proof. Fix $x \in \overline{\nabla}_\infty$. It is enough to show that

$$P_x(X_t \in \nabla_\infty \text{ for all } t \geq s) = 1$$

for every $s > 0$. Let $s > 0$ be arbitrary. By (1.1) and (1.4), we have

$$P_y(X_t \in \nabla_\infty \text{ for all } t \geq 0) = 1 \text{ a.e.-}P(s,x,dy).$$

Therefore,

$$\begin{aligned} P_x(X_t \in \nabla_\infty \text{ for all } t \geq s) &= E_x[P_x(X_t \in \nabla_\infty \text{ for all } t \geq s \mid X_r,\, 0 \leq r \leq s)] \\ &= E_x[P_{X_s}(X_t \in \nabla_\infty \text{ for all } t \geq 0)] \\ &= \int_{\overline{\nabla}_\infty} P_y(X_t \in \nabla_\infty \text{ for all } t \geq 0)\, P(s,x,dy) \\ &= \int_{\overline{\nabla}_\infty} P(s,x,dy) \\ &= 1, \end{aligned}$$

as required. □

Thus, (1.2) is a very simple consequence of two nontrivial results, namely (1.1), which relies on an eigenfunction expansion of the transition density with respect to $\text{PD}_{\alpha,\theta}$, and (1.4), which relies on properties of a Dirichlet form defined in terms of $\text{PD}_{\alpha,\theta}$.

We emphasize that the theorem assumes only $0 \leq \alpha < 1$ (and $\theta > -\alpha$) and therefore includes the special case $\alpha = 0$. The proof of (1.2) by Ethier and Kurtz (1981) in that case may appear more complicated than the proof just given but that is because it is largely self-contained. It does not rely on subtle properties of the diffusion such as (1.1) and (1.4).

Let us conclude by showing why the proof of Ethier and Kurtz fails when $\alpha > 0$. First, we extend the domain of A. Let $\mathscr{H} := \{h \in C^2[0,1] : h(0) = h'(0) = 0\}$, and for $h \in \mathscr{H}$ define $\psi_h \in C(\overline{\nabla}_\infty)$ by

$$\psi_h(x) := \sum_{i=1}^\infty h(x_i).$$

Let A^+ be A acting on the subalgebra of $C(\overline{\nabla}_\infty)$ generated by $\{1\} \cup \{\psi_h : h \in \mathscr{H}\}$. Again, for $\varphi \in \mathscr{D}(A^+)$, $A^+\varphi$ is evaluated on ∇_∞ and extended to $\overline{\nabla}_\infty$ by continuity. It is easy to see that $A^+ \subset \overline{A}$. We also notice that $\varphi_m \in \mathscr{D}(A^+)$ for all real $m \geq 2$ (not just integers), where

$$\varphi_m(x) := \sum_{i=1}^\infty x_i^m, \quad m \in (0,1) \cup (1,\infty), \qquad \varphi_1(x) := 1.$$

This leads to the conclusion that, for every real $m \geq 2$,

$$Z_m(t) := \varphi_m(X_t) - \varphi_m(X_0) - \frac{1}{2}m\int_0^t [(m-1-\alpha)\varphi_{m-1}(X_s) - (m-1+\theta)\varphi_m(X_s)]\,ds$$

is a continuous square-integrable martingale with increasing process

$$I_m(t) = m^2 \int_0^t (\varphi_{2m-1} - \varphi_m^2)(X_s)\,ds.$$

In particular, taking expectations and comparing results for $m = 2$ and $m \to 2+$, we find that

$$P_x(X_t \in \nabla_\infty \text{ for almost all } t > 0) = 1, \quad x \in \overline{\nabla}_\infty.$$

The next step is to extend the conclusion about $Z_m(\cdot)$ to $1 < m < 2$, and this is where the difficulty occurs. Fix such an m and define $h_\varepsilon \in \mathscr{H}$ for $\varepsilon > 0$ by $h_\varepsilon(u) := (u+\varepsilon)^m - \varepsilon^m - m\varepsilon^{m-1}u$. Then $A^+\psi_{h_\varepsilon}$ converges pointwise as $\varepsilon \to 0+$ but not boundedly because of the two terms

$$\frac{1}{2}\sum_{i=1}^\infty x_i h_\varepsilon''(x_i) \quad \text{and} \quad -\frac{1}{2}\alpha\sum_{i=1}^\infty h_\varepsilon'(x_i).$$

Both sums are monotonically increasing as ε decreases to 0, but their coefficients have opposite signs, so the monotone convergence theorem does not apply if $\alpha > 0$, and we cannot show that $\int_0^t \varphi_{m-1}(X_s)\,ds \in L^2$ for all $t > 0$.

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