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## AN EXTENDED HARDY-HILBERT INEQUALITY AND ITS APPLICATIONS

JIA WEIJIAN AND GAO MINGZHE

Department of Mathematics and Computer Science  
Normal College, Jishou University  
Jishou Hunan 416000,  
People's Republic of China.  
EMail: [mingzhegao@163.com](mailto:mingzhegao@163.com)

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Abstract

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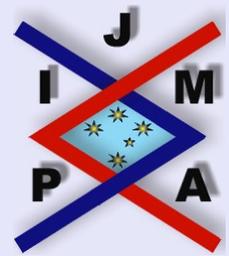


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## Abstract

In this paper, it is shown that an extended Hardy-Hilbert's integral inequality with weights can be established by introducing a power-exponent function of the form  $ax^{1+x}$  ( $a > 0, x \in [0, +\infty)$ ), and the coefficient  $\frac{\pi}{(a)^{1/q}(b)^{1/p} \sin \pi/p}$  is shown to be the best possible constant in the inequality. In particular, for the case  $p = 2$ , some extensions on the classical Hilbert's integral inequality are obtained. As applications, generalizations of Hardy-Littlewood's integral inequality are given.

*2000 Mathematics Subject Classification:* 26D15.

*Key words:* Power-exponent function, Weight function, Hardy-Hilbert's integral inequality, Hardy-Littlewood's integral inequality.

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# 1. Introduction

The famous Hardy-Hilbert's integral inequality is

$$(1.1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \\ \leq \frac{\pi}{\sin \frac{\pi}{p}} \left\{ \int_0^\infty f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(y) dy \right\}^{\frac{1}{q}},$$

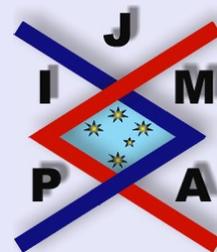
where  $p > 1$ ,  $q = p/(p-1)$  and the constant  $\frac{\pi}{\sin \frac{\pi}{p}}$  is best possible (see [1]). In particular, when  $p = q = 2$ , the inequality (1.1) is reduced to the classical Hilbert integral inequality:

$$(1.2) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \pi \left\{ \int_0^\infty f^2(x) dx \right\}^{\frac{1}{2}} \left\{ \int_0^\infty g^2(y) dy \right\}^{\frac{1}{2}},$$

where the coefficient  $\pi$  is best possible.

Recently, the following result was given by introducing the power function in [2]:

$$(1.3) \quad \int_a^b \int_a^b \frac{f(x)g(y)}{x^t + y^t} dx dy \\ \leq \left\{ \omega(t, p, q) \int_a^b x^{1-t} f^p(x) dx \right\}^{\frac{1}{p}} \\ \times \left\{ \omega(t, q, p) \int_a^b x^{1-t} g^q(x) dx \right\}^{\frac{1}{q}},$$



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where  $t$  is a parameter which is independent of  $x$  and  $y$ ,  $\omega(t, p, q) = \frac{\pi}{t \sin \frac{\pi}{pt}} - \varphi(q)$  and here the function  $\varphi$  is defined by

$$\varphi(r) = \int_0^{a/b} \frac{u^{t-2+1/r}}{1+u^t} du, \quad r = p, q.$$

However, in [2] the best constant for (1.3) was not determined.

Afterwards, various extensions on the inequalities (1.1) and (1.2) have appeared in some papers (such as [3, 4] etc.). The purpose of the present paper is to show that if the denominator  $x + y$  of the function on the left-hand side of (1.1) is replaced by the power-exponent function  $ax^{1+x} + by^{1+y}$ , then we can obtain a new inequality and show that the coefficient  $\frac{\pi}{(a)^{1/q}(b)^{1/p} \sin \pi/p}$  is the best constant in the new inequality. In particular if  $p = 2$  then several extensions of (1.2) follow. As its applications, it is shown that extensions on the Hardy-Littlewood integral inequality can be established.

Throughout this paper we stipulate that  $a > 0$  and  $b > 0$ .

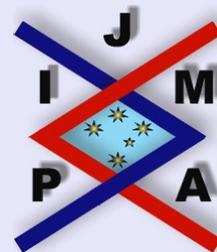
For convenience, we give the following lemma which will be used later.

**Lemma 1.1.** *Let  $h(x) = \frac{x}{1+x+\ln x}$ ,  $x \in (0, +\infty)$ , then there exists a function  $\varphi(x)$  ( $0 \leq \varphi(x) < \frac{1}{2}$ ), such that  $h(x) = \frac{1}{2} - \varphi(x)$ .*

*Proof.* Consider the function defined by

$$s(x) = \frac{1+x}{x} + \ln x, \quad x \in (0, +\infty).$$

It is easy to see that the minimum of  $s(x)$  is 2. Hence  $s(x) \geq 2$ , and  $h(x) = s^{-1}(x) \leq \frac{1}{2}$ . Obviously  $h(x) = \frac{1}{s(x)} > 0$ . We can define a nonnegative function




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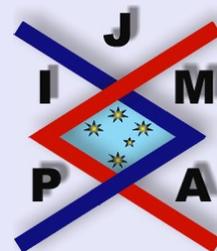
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$\varphi$  by

$$(1.4) \quad \varphi(x) = \frac{1 - x + x \ln x}{2(1 + x + x \ln x)} \quad x \in (0, +\infty).$$

Hence we have  $h(x) = \frac{1}{2} - \varphi(x)$ . The lemma follows. □



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## 2. Main Results

Define a function by

$$(2.1) \quad \omega(r, x) = x^{(1+x)(1-r)} \left( \frac{1}{2} - \varphi(x) \right)^{r-1} \quad x \in (0, +\infty),$$

where  $r > 1$  and  $\varphi(x)$  is defined by (1.4).

**Theorem 2.1.** *Let*

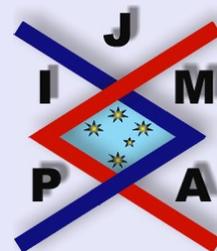
$$0 < \int_0^\infty \omega(p, x) f^p(x) dx < +\infty, \quad 0 < \int_0^\infty \omega(q, x) g^q(x) dx < +\infty,$$

*the weight function  $\omega(r, x)$  is defined by (2.1),  $\frac{1}{p} + \frac{1}{q} = 1$ , and  $p \geq q > 1$ . Then*

$$(2.2) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{ax^{1+x} + by^{1+y}} dx dy \leq \frac{\mu\pi}{\sin \frac{\pi}{p}} \left\{ \int_0^\infty \omega(p, x) f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \omega(q, x) g^q(x) dx \right\}^{\frac{1}{q}},$$

where  $\mu = (1/a)^{1/q} (1/b)^{1/p}$  and the constant factor  $\frac{\mu\pi}{\sin \frac{\pi}{p}}$  is best possible.

*Proof.* Let  $f(x) = F(x) \{(ax^{1+x})'\}^{\frac{1}{q}}$  and  $g(y) = G(y) \{(by^{1+y})'\}^{\frac{1}{p}}$ . Define



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two functions by

$$(2.3) \quad \alpha = \frac{F(x) \{(by^{1+y})'\}^{\frac{1}{p}}}{(ax^{1+x} + by^{1+y})^{\frac{1}{p}}} \left( \frac{ax^{1+x}}{by^{1+y}} \right)^{\frac{1}{pq}} \quad \text{and}$$

$$\beta = \frac{G(y) \{(ax^{1+x})'\}^{\frac{1}{q}}}{(ax^{1+x} + by^{1+y})^{\frac{1}{q}}} \left( \frac{by^{1+y}}{ax^{1+x}} \right)^{\frac{1}{pq}}.$$

Let us apply Hölder's inequality to estimate the right hand side of (2.2) as follows:

$$(2.4) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{ax^{1+x} + by^{1+y}} dx dy$$

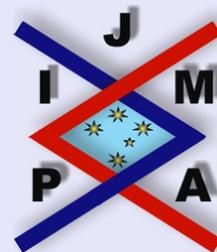
$$= \int_0^\infty \int_0^\infty \alpha \beta dx dy$$

$$\leq \left\{ \int_0^\infty \int_0^\infty \alpha^p dx dy \right\}^{\frac{1}{p}} \left\{ \int_0^\infty \int_0^\infty \beta^q dx dy \right\}^{\frac{1}{q}}.$$

It is easy to deduce that

$$\int_0^\infty \int_0^\infty \alpha^p dx dy = \int_0^\infty \int_0^\infty \frac{(by^{1+y})'}{ax^{1+x} + by^{1+y}} \left( \frac{ax^{1+x}}{by^{1+y}} \right)^{\frac{1}{q}} F^p(x) dx dy$$

$$= \int_0^\infty \omega_q F^p(x) dx.$$



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We compute the weight function  $\omega_q$  as follows:

$$\begin{aligned}\omega_q &= \int_0^\infty \frac{(by^{1+y})'}{ax^{1+x} + by^{1+y}} \left(\frac{ax^{1+x}}{by^{1+y}}\right)^{\frac{1}{q}} dy \\ &= \int_0^\infty \frac{1}{ax^{1+x} + by^{1+y}} \left(\frac{ax^{1+x}}{by^{1+y}}\right)^{\frac{1}{q}} d(by^{1+y}).\end{aligned}$$

Let  $t = by^{1+y}/ax^{1+x}$ . Then we have

$$\omega_q = \int_0^\infty \frac{1}{1+t} \left(\frac{1}{t}\right)^{\frac{1}{q}} dt = \frac{\pi}{\sin \frac{\pi}{q}} = \frac{\pi}{\sin \frac{\pi}{p}}.$$

Notice that  $F(x) = \{(ax^{1+x})'\}^{-1/q} f(x)$ . Hence we have

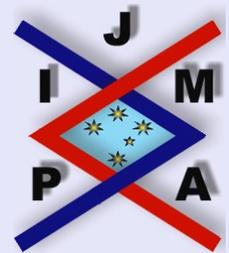
$$(2.5) \quad \int_0^\infty \int_0^\infty \alpha^p dx dy = \frac{\pi}{\sin \frac{\pi}{p}} \int_0^\infty \left((ax^{1+x})'\right)^{1-p} f^p(x) dx,$$

and, similarly,

$$(2.6) \quad \int_0^\infty \int_0^\infty \beta^q dx dy = \frac{\pi}{\sin \frac{\pi}{p}} \int_0^\infty \left((by^{1+y})'\right)^{1-q} g^q(y) dy.$$

Substituting (2.5) and (2.6) into (2.3), we obtain

$$(2.7) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{ax^{1+x} + by^{1+y}} dx dy$$



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$$\leq \frac{\pi}{\sin \frac{\pi}{p}} \left\{ \int_0^{\infty} \left( (ax^{1+x})' \right)^{1-p} f^p(x) dx \right\}^{\frac{1}{p}} \\ \times \left\{ \int_0^{\infty} \left( (by^{1+y})' \right)^{1-q} g^q(y) dy \right\}^{\frac{1}{q}}.$$

We need to show that the constant factor  $\frac{\pi}{\sin \frac{\pi}{p}}$  contained in (2.7) is best possible.

Define two functions by

$$\tilde{f}(x) = \begin{cases} 0, & x \in (0, 1) \\ (ax^{1+x})^{-(1+\varepsilon)/p} (ax^{1+x})', & x \in [1, +\infty) \end{cases}$$

and

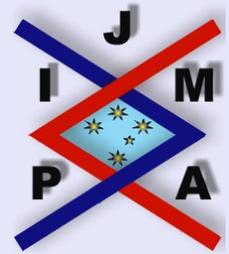
$$\tilde{g}(y) = \begin{cases} 0, & y \in (0, 1) \\ (by^{1+y})^{-(1+\varepsilon)/q} (by^{1+y})', & y \in [1, +\infty) \end{cases}.$$

Assume that  $0 < \varepsilon < \frac{q}{2p}$  ( $p \geq q > 1$ ). Then

$$\int_0^{+\infty} \left( (ax^{1+x})' \right)^{1-p} \tilde{f}^p(x) dx = \int_1^{+\infty} (ax^{1+x})^{-1-\varepsilon} d(ax^{1+x}) = \frac{1}{\varepsilon}.$$

Similarly, we have

$$\int_0^{\infty} \left( (by^{1+y})' \right)^{1-q} \tilde{g}^q(y) dy = \frac{1}{\varepsilon}.$$



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If  $\frac{\pi}{\sin \frac{\pi}{p}}$  is not best possible, then there exists  $k > 0$ ,  $k < \frac{\pi}{\sin \frac{\pi}{p}}$  such that

$$(2.8) \quad \int_0^\infty \int_0^\infty \frac{\tilde{f}(x) \tilde{g}(y)}{ax^{1+x} + by^{1+y}} dx dy$$

$$< k \left( \int_0^\infty \left( (ax^{1+x})' \right)^{1-p} \tilde{f}^p(x) dx \right)^{\frac{1}{p}}$$

$$\times \left( \int_0^\infty \left( (by^{1+y})' \right)^{1-q} \tilde{g}^q(y) dy \right)^{\frac{1}{q}} = \frac{k}{\varepsilon}.$$

On the other hand, we have

$$\int_0^\infty \int_0^\infty \frac{\tilde{f}(x) \tilde{g}(y)}{ax^{1+x} + by^{1+y}} dx dy$$

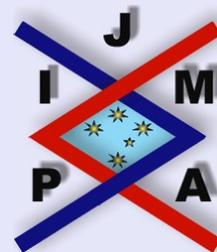
$$= \int_1^\infty \int_1^\infty \frac{\left\{ (ax^{1+x})^{-\frac{1+\varepsilon}{p}} (ax^{1+x})' \right\} \left\{ (by^{1+y})^{-\frac{1+\varepsilon}{q}} (by^{1+y})' \right\}}{ax^{1+x} + by^{1+y}} dx dy$$

$$= \int_1^\infty \left\{ \int_1^\infty \frac{(by^{1+y})^{-\frac{1+\varepsilon}{q}}}{ax^{1+x} + by^{1+y}} d(by^{1+y}) \right\} \left\{ (ax^{1+x})^{-\frac{1+\varepsilon}{p}} (ax^{1+x})' \right\} dx$$

$$= \int_1^\infty \left\{ \int_{b/ax^{1+x}}^\infty \frac{1}{1+t} \left( \frac{1}{t} \right)^{-\frac{1+\varepsilon}{q}} dt \right\} (ax^{1+x})^{-1-\varepsilon} d(ax^{1+x})$$

$$= \frac{1}{\varepsilon} \int_{b/ax^{1+x}}^\infty \frac{1}{1+t} \left( \frac{1}{t} \right)^{-\frac{1+\varepsilon}{q}} dt.$$

If the lower limit  $b/ax^{1+x}$  of this integral is replaced by zero, then the resulting



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error is smaller than  $\frac{(b/ax^{1+x})^\alpha}{\alpha}$ , where  $\alpha$  is positive and independent of  $\varepsilon$ . In fact, we have

$$\int_0^{b/ax^{1+x}} \frac{1}{1+t} \left(\frac{1}{t}\right)^{\frac{1+\varepsilon}{q}} dt < \int_0^{b/ax^{1+x}} t^{-(1+\varepsilon)/q} dt = \frac{(b/ax^{1+x})^\beta}{\beta}$$

where  $\beta = 1 - (1 + \varepsilon)/q$ . If  $0 < \varepsilon < \frac{q}{2p}$ , then we may take  $\alpha$  such that

$$\alpha = 1 - \frac{1 + q/2p}{q} = \frac{1}{2p}.$$

Consequently, we get

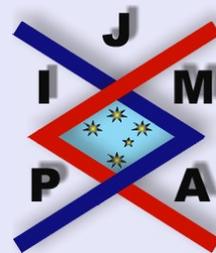
$$(2.9) \quad \int_0^\infty \int_0^\infty \frac{\tilde{f}(x) \tilde{g}(y)}{ax^{1+x} + by^{1+y}} dx dy > \frac{1}{\varepsilon} \left\{ \frac{\pi}{\sin \frac{\pi}{p}} + o(1) \right\} \quad (\varepsilon \rightarrow 0).$$

Clearly, when  $\varepsilon$  is small enough, the inequality (2.7) is in contradiction with (2.9). Therefore,  $\frac{\pi}{\sin \frac{\pi}{p}}$  is the best possible value for which the inequality (2.7) is valid.

Let  $u = ax^{1+x}$  and  $v = by^{1+y}$ . Then

$$u' = ax^{1+x} \left( \frac{1+x}{x} + \ln x \right) = ax^{1+x} h^{-1}(x).$$

Similarly, we have  $v' = by^{1+y} h^{-1}(y)$ . Substituting them into (2.7) and then using Lemma 1.1, the inequality (2.2) yields after simplifications. The constant factor  $\frac{\mu\pi}{\sin \frac{\pi}{p}}$  is best possible, where  $\mu = (1/a)^{1/q} (1/b)^{1/p}$ . Thus the proof of the theorem is completed.  $\square$



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It is known from (2.1) that

$$\omega(r, x) = x^{(1+x)(1-r)} \left( \frac{1}{2} - \varphi(x) \right)^{r-1} = \left( \frac{1}{2} \right)^{r-1} x^{(1+x)(1-r)} (1 - 2\varphi(x))^{r-1}.$$

The following result is equivalent to Theorem 2.1.

**Theorem 2.2.** Let  $\varphi(x)$  be a function defined by (1.4),  $\frac{1}{p} + \frac{1}{q} = 1$  and  $p \geq q > 1$ .  
If

$$0 < \int_0^\infty x^{(1+x)(1-p)} (1 - 2\varphi(x))^{p-1} f^p(x) dx < +\infty \quad \text{and}$$

$$0 < \int_0^\infty y^{(1+y)(1-q)} (1 - 2\varphi(y))^{q-1} g^q(y) dy < +\infty,$$

then

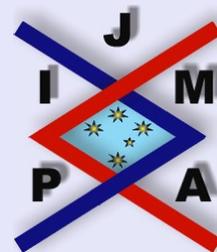
$$(2.10) \quad \int_0^\infty \int_0^\infty \frac{f(x) g(y)}{ax^{1+x} + by^{1+y}} dx dy$$

$$\leq \frac{\mu\pi}{2 \sin \frac{\pi}{p}} \left\{ \int_0^\infty x^{(1+x)(1-p)} (1 - 2\varphi(x))^{p-1} f^p(x) dx \right\}^{\frac{1}{p}}$$

$$\times \left\{ \int_0^\infty y^{(1+y)(1-q)} (1 - 2\varphi(y))^{q-1} g^q(y) dy \right\}^{\frac{1}{q}},$$

where  $\mu = (1/a)^{1/q} (1/b)^{1/p}$  and the constant factor  $\frac{\mu\pi}{2 \sin \frac{\pi}{p}}$  is best possible.

In particular, for case  $p = 2$ , some extensions on (1.2) are obtained. According to Theorem 2.1, we get the following results.



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**Corollary 2.3.** *If*

$$0 < \int_0^\infty x^{-(1+x)} \left( \frac{1}{2} - \varphi(x) \right) f^2(x) dx < +\infty \quad \text{and}$$

$$0 < \int_0^\infty y^{-(1+y)} \left( \frac{1}{2} - \varphi(y) \right) g^2(y) dy < +\infty,$$

where  $\varphi(x)$  is a function defined by (1.4), then

$$(2.11) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{ax^{1+x} + by^{1+y}} dx dy$$

$$\leq \frac{\pi}{\sqrt{ab}} \left\{ \int_0^\infty x^{-(1+x)} \left( \frac{1}{2} - \varphi(x) \right) f^2(x) dx \right\}^{\frac{1}{2}}$$

$$\times \left\{ \int_0^\infty y^{-(1+y)} \left( \frac{1}{2} - \varphi(y) \right) g^2(y) dy \right\}^{\frac{1}{2}},$$

where the constant factor  $\frac{\pi}{\sqrt{ab}}$  is best possible.

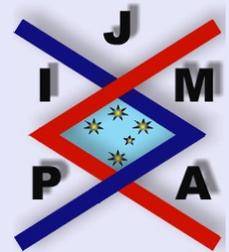
**Corollary 2.4.** *Let  $\varphi(x)$  be a function defined by (1.4). If*

$$0 < \int_0^\infty x^{-(1+x)} \left( \frac{1}{2} - \varphi(x) \right) f^2(x) dx < +\infty,$$

then

$$(2.12) \quad \int_0^\infty \int_0^\infty \frac{f(x)f(y)}{ax^{1+x} + by^{1+y}} dx dy$$

$$\leq \frac{\pi}{\sqrt{ab}} \int_0^\infty x^{-(1+x)} \left( \frac{1}{2} - \varphi(x) \right) f^2(x) dx,$$



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where the constant factor  $\frac{\pi}{\sqrt{ab}}$  is best possible.

A equivalent proposition of Corollary 2.3 is:

**Corollary 2.5.** Let  $\varphi(x)$  be a function defined by (1.4),

$$0 < \int_0^{\infty} x^{-(1+x)} (1 - 2\varphi(x)) f^2(x) dx < +\infty \quad \text{and}$$

$$0 < \int_0^{\infty} y^{-(1+y)} (1 - 2\varphi(y)) g^2(y) dy < +\infty,$$

then

$$(2.13) \quad \int_0^{\infty} \int_0^{\infty} \frac{f(x) g(y)}{ax^{1+x} + by^{1+y}} dx dy$$

$$\leq \frac{\pi}{2\sqrt{ab}} \left\{ \int_0^{\infty} x^{-(1+x)} (1 - 2\varphi(x)) f^2(x) dx \right\}^{\frac{1}{2}}$$

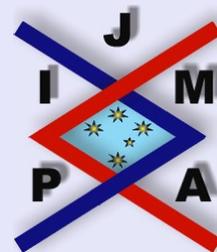
$$\times \left\{ \int_0^{\infty} y^{-(1+y)} (1 - 2\varphi(y)) g^2(y) dy \right\}^{\frac{1}{2}},$$

where the constant factor  $\frac{\pi}{2\sqrt{ab}}$  is best possible.

Similarly, an equivalent proposition to Corollary 2.4 is:

**Corollary 2.6.** Let  $\varphi(x)$  be a function defined by (1.4). If

$$0 < \int_0^{\infty} x^{-(1+x)} (1 - 2\varphi(x)) f^2(x) dx < +\infty,$$



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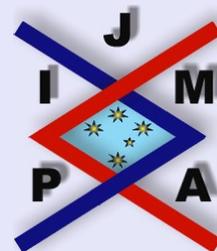
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then

$$(2.14) \quad \int_0^\infty \int_0^\infty \frac{f(x) f(y)}{ax^{1+x} + by^{1+y}} dx dy \leq \frac{\pi}{2\sqrt{ab}} \int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x) dx,$$

where the constant factor  $\frac{\pi}{2\sqrt{ab}}$  is best possible.



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### 3. Application

In this section, we will give various extensions of Hardy-Littlewood's integral inequality.

Let  $f(x) \in L^2(0, 1)$  and  $f(x) \neq 0$ . If

$$a_n = \int_0^1 x^n f(x) dx, \quad n = 0, 1, 2, \dots$$

then we have the Hardy-Littlewood's inequality (see [1]) of the form

$$(3.1) \quad \sum_{n=0}^{\infty} a_n^2 < \pi \int_0^1 f^2(x) dx,$$

where  $\pi$  is the best constant that keeps (3.1) valid. In our previous paper [5], the inequality (3.1) was extended and the following inequality established:

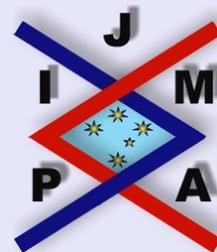
$$(3.2) \quad \int_0^{\infty} f^2(x) dx < \pi \int_0^1 h^2(x) dx,$$

where  $f(x) = \int_0^1 t^x h(t) dt$ ,  $x \in [0, +\infty)$ .

Afterwards the inequality (3.2) was refined into the form in the paper [6]:

$$(3.3) \quad \int_0^{\infty} f^2(x) dx \leq \pi \int_0^1 th^2(t) dt.$$

We will further extend the inequality (3.3), some new results can be obtained by further extending inequality (3.3).



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**Theorem 3.1.** Let  $h(t) \in L^2(0, 1)$ ,  $h(t) \neq 0$ . Define a function by

$$f(x) = \int_0^1 t^{u(x)} |h(t)| dt, \quad x \in [0, +\infty),$$

where  $u(x) = x^{1+x}$ . Also, let  $\varphi(x)$  be a weight function defined by (1.4), ( $r = p, q$ ),  $\frac{1}{p} + \frac{1}{q} = 1$  and  $p \geq q > 1$ . If

$$0 < \int_0^\infty x^{(1+x)(1-r)} \left(\frac{1}{2} - \varphi(x)\right)^{r-1} f^r(x) dx < +\infty,$$

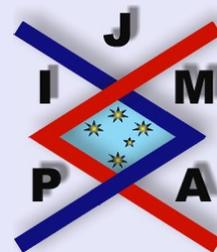
then

$$(3.4) \quad \left(\int_0^\infty f^2(x) dx\right)^2 < \frac{\mu\pi}{\sin \frac{\pi}{p}} \left(\int_0^\infty x^{(1+x)(1-p)} \left(\frac{1}{2} - \varphi(x)\right)^{p-1} f^p(x) dx\right)^{\frac{1}{p}} \times \left(\int_0^\infty y^{(1+y)(1-q)} \left(\frac{1}{2} - \varphi(y)\right)^{q-1} f^q(y) dy\right)^{\frac{1}{q}} \int_0^1 th^2(t) dt,$$

where the constant factor  $\frac{\mu\pi}{\sin \frac{\pi}{p}}$  in (3.4) is best possible, and  $\mu = (1/a)^{1/q} (1/b)^{1/p}$ .

*Proof.* Let us write  $f^2(x)$  in the form:

$$f^2(x) = \int_0^1 f(x) t^{u(x)} |h(t)| dt.$$



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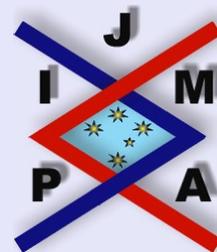
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We apply, in turn, Schwarz's inequality and Theorem 2.1 to obtain

$$\begin{aligned}
 (3.5) \quad & \left( \int_0^\infty f^2(x) dx \right)^2 \\
 &= \left\{ \int_0^\infty \left( \int_0^1 f(x) t^{u(x)} |h(t)| dt \right) dx \right\}^2 \\
 &= \left\{ \int_0^1 \left( \int_0^\infty f(x) t^{u(x)-1/2} dx \right) t^{1/2} |h(t)| dt \right\}^2 \\
 &\leq \int_0^1 \left( \int_0^\infty f(x) t^{u(x)-1/2} dx \right)^2 dt \int_0^1 th^2(t) dt \\
 &= \int_0^1 \left( \int_0^\infty f(x) t^{u(x)-1/2} dx \right) \\
 &\quad \times \left( \int_0^\infty f(y) t^{u(y)-1/2} dy \right) dt \int_0^1 th^2(t) dt \\
 &= \int_0^1 \left( \int_0^\infty \int_0^\infty f(x) f(y) t^{u(x)+u(y)-1} dx dy \right) dt \int_0^1 th^2(t) dt \\
 &= \left( \int_0^\infty \int_0^\infty \frac{f(x) f(y)}{u(x) + u(y)} dx dy \right) \int_0^1 th^2(t) dt \\
 &\leq \frac{\mu\pi}{\sin \frac{\pi}{p}} \left\{ \int_0^\infty x^{(1+x)(1-p)} \left( \frac{1}{2} - \varphi(x) \right)^{p-1} f^p(x) dx \right\}^{\frac{1}{p}} \\
 &\quad \times \left\{ \int_0^\infty y^{(1+y)(1-q)} \left( \frac{1}{2} - \varphi(y) \right)^{q-1} f^q(y) dy \right\}^{\frac{1}{q}} \int_0^1 th^2(t) dt.
 \end{aligned}$$



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Since  $h(t) \neq 0$ ,  $f^2(x) \neq 0$ . It is impossible to take equality in (3.5). We therefore complete the proof of the theorem.  $\square$

An equivalent proposition to Theorem 3.1 is:

**Theorem 3.2.** *Let the functions  $h(t)$ ,  $f(x)$  and  $u(x)$  satisfy the assumptions of Theorem 3.1, and assume that*

$$0 < \int_0^\infty x^{(1+x)(1-r)} (1 - 2\varphi(x))^{r-1} f^r(x) dx < +\infty \quad (r = p, q).$$

Then

$$(3.6) \quad \left( \int_0^\infty f^2(x) dx \right)^2 < \frac{\mu\pi}{2 \sin \frac{\pi}{p}} \left( \int_0^\infty x^{(1+x)(1-p)} (1 - 2\varphi(x))^{p-1} f^p(x) dx \right)^{\frac{1}{p}} \times \left( \int_0^\infty y^{(1+y)(1-q)} (1 - 2\varphi(y))^{q-1} f^q(y) dy \right)^{\frac{1}{q}} \int_0^1 th^2(t) dt,$$

and the constant factor  $\frac{\mu\pi}{\sin \frac{\pi}{p}}$  in (3.6) is best possible, where  $\mu = (1/a)^{1/q} (1/b)^{1/p}$ .

In particular, when  $p = q = 2$ , we have the following result.

**Corollary 3.3.** *Let the functions  $h(t)$ ,  $f(x)$  and  $u(x)$  satisfy the assumptions of Theorem 3.1, and assume that*

$$0 < \int_0^\infty x^{-(1+x)} \left( \frac{1}{2} - \varphi(x) \right) f^2(x) dx < +\infty,$$



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where  $\varphi(x)$  is a function defined by (1.4). Then

$$(3.7) \quad \left( \int_0^\infty f^2(x) dx \right)^2 < \frac{\pi}{\sqrt{ab}} \left( \int_0^\infty x^{-(1+x)} \left( \frac{1}{2} - \varphi(x) \right) f^2(x) dx \right) \int_0^1 th^2(t) dt,$$

and the constant factor  $\frac{\pi}{\sqrt{ab}}$  in (3.7) is best possible.

A result equivalent to Corollary 3.3 is:

**Corollary 3.4.** *Let the functions  $h(t)$ ,  $f(x)$  and  $u(x)$  satisfy the assumptions of Theorem 3.1, and assume that*

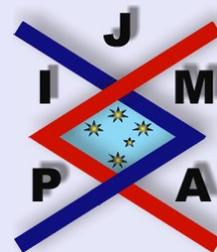
$$0 < \int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x) dx < +\infty,$$

where  $\varphi(x)$  is a function defined by (1.4). Then

$$(3.8) \quad \left( \int_0^\infty f^2(x) dx \right)^2 < \frac{\pi}{2\sqrt{ab}} \left( \int_0^\infty x^{-(1+x)} (1 - 2\varphi(x)) f^2(x) dx \right) \int_0^1 th^2(t) dt,$$

and the constant factor  $\frac{\pi}{2\sqrt{ab}}$  in (3.8) is best possible.

The inequalities (3.4), (3.6), (3.7) and (3.8) are extensions of (3.3).



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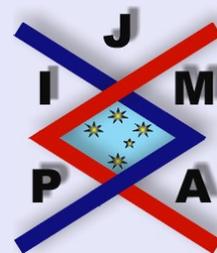
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