S-GEOMETRIC CONVEXITY OF A FUNCTION INVOLVING MACLAURIN'S ELEMENTARY SYMMETRIC MEAN

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Abstract: Let $x_i > 0, i = 1, 2, \dots, n$, $x = (x_1, x_2, \dots, x_n)$, the kth elementary symmetric function of x is defined as $E_n(x, k) = \sum_{1 \le i_1 < \cdots < i_k \le n} \prod_{j=1}^{k} x_{i_j}$ with $1 \le k \le n$, the kth elementary symmetric mean is defined as $P_n(x, k) = \left(\binom{n}{k}^{-1} E_n(x, k)\right)^{\frac{1}{k}}$, and the function f is defined as $f(x) = P_n(x, k-1) - P_n(x, k)$. The paper proves that f is a S-geometrically convex function. The result generalizes the well-known Maclaurin-Inequality.

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1. Introduction

Throughout the paper we assume $\mathbb{R}^n$ be the n-dimensional Euclidean Space,

$$\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n), x_i > 0, i = 1, 2, \ldots, n\},$$

and

$$e^x = (e^{x_1}, e^{x_2}, \ldots, e^{x_n}), \quad x^c = (x_1^c, x_2^c, \ldots, x_n^c),$$

$$\ln x = (\ln x_1, \ln x_2, \ldots, \ln x_n), x \cdot y = (x_1 y_1, x_2 y_2, \ldots, x_n y_n),$$

where $c \in \mathbb{R}$, and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. And if $n \geq 2$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$. They are defined respectively by

$$A_n(x) = \frac{x_1 + x_2 + \cdots + x_n}{n}, \quad G_n(a) = \sqrt[n]{x_1 x_2 \cdots x_n}.$$

The kth elementary symmetric function of x, kth elementary symmetric mean, and function f are defined respectively as

$$E_n(x, k) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \prod_{j=1}^{k} x_{i_j}, \qquad (1 \leq k \leq n)$$

$$P_n(x, k) = \left(\binom{n}{k}^{-1} E_n(x, k) \right)^{\frac{1}{k}}, \qquad (1 \leq k \leq n)$$

$$f(x) = P_n(x, k-1) - P_n(x, k), \qquad (2 \leq k \leq n)$$

with $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

The following theorem is true by [1].

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Theorem 1.1 (Maclaurin-Inequality).

$$A_n(x) = P_n(x,1) \geq P_n(x,2) \geq \cdots \geq P_n(x,n-1) \geq P_n(x,n) = G_n(x). \tag{1.1}$$

$$\frac{E_n(x,n)}{E_n(x,n-1)} < \frac{E_n(x,n-1)}{E_n(x,n-2)} < \cdots < \frac{E_n(x,3)}{E_n(x,2)} < \frac{E_n(x,2)}{E_n(x,1)} < E_n(x,1). \tag{1.2}$$

References [5], [4], [2], [7], [8], [9], [10] and [6] give the definitions of n dimensional geometrically convex functions, S-geometrically convex functions and logarithm majorization, and a large number of results have been obtained. Since many functions have geometric convexity or geometric concavity, research into geometrically convex functions make sense. For a comprehensive list of recent results on geometrically convex functions, see the book [10] and the papers [5], [4], [2] [7] [8], [9] and [6] where further results are given.

The main aim of this paper is to prove the following theorem.

Theorem 1.2. *Let $n = 2$, or $n \geq 3, 2 \leq k-1 \leq n-1$, and $f(x) = P_n(x,k-1) - P_n(x,k)$, then f is a $S-$geometrically convex function.*

The result generalizes the Maclaurin-Inequality.

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2. Relative Definition and a Lemma

Lemma 2.1 ([3]). *Let $H \subseteq \mathbb{R}^n$ be a symmetric convex set with a nonempty interior, $\phi : H \to \mathbb{R}$ be continuously differentiable on the interior of H and continuous on H. Necessary and sufficient conditions for ϕ to be S-convex(concave) on H are that ϕ is symmetric on H, and*

$$(x_1 - x_2)\left(\frac{\partial \phi}{\partial x_1} - \frac{\partial \phi}{\partial x_2}\right) \geq (\leq)\, 0,$$

for all x in the interior of H.

Definition 2.1 ([9], [10, p. 89], [6]). *Let $x \in \mathbb{R}^n_+$, $y \in \mathbb{R}^n_+$, $\left(x_{[1]}, x_{[2]}, \ldots, x_{[n]}\right)$ and $\left(y_{[1]}, y_{[2]}, \ldots, y_{[n]}\right)$ be the decreasing queues of $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ respectively. We say $(x_1, x_2, \ldots, x_n)$ logarithm majorizes $(y_1, y_2, \ldots, y_n)$, denote $\ln x \succ \ln y$ if*

$$\begin{cases} \prod_{i=1}^{k} x_i \geq \prod_{i=1}^{k} y_i, \quad k = 1, 2, \ldots, n-1, \\ \prod_{i=1}^{n} x_i = \prod_{i=1}^{n} y_i. \end{cases}$$

Remark 1. If x logarithm majorizes y, then

$$\begin{cases} \ln\left(\prod_{i=1}^{k} x_i\right) \geq \ln\left(\prod_{i=1}^{k} y_i\right), \quad k = 1, 2, \ldots, n-1, \\ \ln\left(\prod_{i=1}^{n} x_i\right) = \ln\left(\prod_{i=1}^{n} y_i\right). \end{cases}$$

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$$\begin{cases} \sum_{i=1}^{k} \ln x_i \geq \sum_{i=1}^{k} \ln y_i, \quad k = 1, 2, \dots, n-1, \\ \\ \sum_{i=1}^{n} \ln x_i = \sum_{i=1}^{n} \ln y_i. \end{cases}$$

So we can denote $\ln x \succ \ln y$ in the Definition 2.1.

Lemma 2.2 ([10, p. 97]). $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}_+^n$ *logarithm majorizes*

$$\bar{G}(x) = (G_n(x), G_n(x), \dots, G_n(x)).$$

Definition 2.2 ([7]). *Let $E \subseteq \mathbb{R}_+^n$, then E is said to be a logarithm convex set, if for any $x, y \in E$, $\alpha, \beta > 0, \alpha + \beta = 1$, it have $x^\alpha y^\beta \in E$.*

Remark 2. Let $E \subseteq \mathbb{R}_+^n$, $\ln E = \{\ln x | x \in E\}$. Then $x, y \in E$ if only if $\ln x, \ln y \in \ln E$, and $x^\alpha y^\beta \in E$ if only if $\alpha \ln x + \beta \ln y \in \ln E$, so E is a logarithm convex set if and only if $\ln E$ is a convex set.

Definition 2.3 ([10, p. 107]). *Let $E \subseteq \mathbb{R}_+^n$, $f : E \to [0, +\infty)$. Then f is called an $S-$geometrically convex function, if for any $x, y \in E \subseteq \mathbb{R}_+^n$, $\ln x \succ \ln y$, we have*

$$f(x) \geq f(y). \tag{2.1}$$

And f is called an $S-$geometrically concave function, if the inequality (2.1) is reversed.

Lemma 2.3 ([10, p. 108]). *Let $E \subseteq \mathbb{R}_+^n$ be a symmetric logarithm convex set with a nonempty interior, $f : E \to [0, +\infty)$ be symmetric continuously differentiable on the interior of E and continuous on E. Then f is a S-geometrically convex function, if the following inequality*

$$(\ln x_1 - \ln x_2)\left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}\right) \geq 0 \tag{2.2}$$

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holds for all x in the interior of E. And f a is S-geometrically concave function, if the inequality (2.2) is reversed.

Proof. Let $\ln E = \{\ln x | x \in E\}$, then $\ln E$ is a symmetric convex set and has a nonempty interior. Again, let $\varepsilon > 0$, $g(x) = f(x) + \varepsilon$ with $x \in E$, and $h(y) = \ln g(e^y)$ with $y \in \ln E = \{\ln x | x \in E\}$, then $g : E \to (0, +\infty)$, $h : \ln E \to (-\infty, +\infty)$. Further let $x = e^y$,

$$\begin{aligned}\frac{\partial h}{\partial y_1} &= \frac{\partial(\ln g(e^y))}{\partial y_1} \\ &= \frac{1}{g(e^y)} \cdot \frac{\partial(g(e^y))}{\partial y_1} \\ &= \frac{1}{g(x)} \cdot \frac{\partial(g(x))}{\partial x_1} \cdot e^{y_1} \\ &= \frac{x_1}{g(x)} \cdot \frac{\partial g}{\partial x_1}.\end{aligned}$$

Similarly,

$$\frac{\partial h}{\partial y_2} = \frac{x_2}{g(x)} \cdot \frac{\partial g}{\partial x_2}.$$

According to inequality 2.2,

$$\begin{aligned}(y_1 - y_2)\left(\frac{\partial h}{\partial y_1} - \frac{\partial h}{\partial y_2}\right) &= \frac{(\ln x_1 - \ln x_2)}{g(x)}\left(x_1 \frac{\partial g}{\partial x_1} - x_2 \frac{\partial g}{\partial x_2}\right) \\ &= \frac{(\ln x_1 - \ln x_2)}{g(x)}\left(x_1 \frac{\partial f}{\partial x_1} - x_2 \frac{\partial f}{\partial x_2}\right) \geq 0.\end{aligned}$$

Then by Lemma 2.1, we know that h is a S-convex function. For any $u, v \in E$ with

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$\ln u \succ \ln v$, we have

$$h(\ln u) \geq h(\ln v), \quad \ln g\left(e^{\ln u}\right) \geq \ln g\left(e^{\ln v}\right),$$

and

$$g(u) \geq g(v), \quad f(u) \geq f(v).$$

So f is a S- geometrically convex function.

If the inequality (2.2) is reversed, we similarly have that f is a S-geometrically concave function.

The proof of Lemma 2.3 is completed. □

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3. The Proof of Theorem 1.2

Proof of Theorem 1.2. If $n=2$, then $k=2$.

$$f(x)=P_n(x,k-1)-P_n(x,k)=\frac{x_1+x_2}{2}-\sqrt{x_1x_2},$$

$$\frac{\partial f}{\partial x_1}=\frac{1}{2}-\frac{1}{2}\sqrt{\frac{x_2}{x_1}},\qquad x_1\frac{\partial f}{\partial x_1}=\frac{1}{2}x_1-\frac{1}{2}\sqrt{x_1x_2},$$

$$(\ln x_1-\ln x_2)\left(x_1\frac{\partial f}{\partial x_1}-x_2\frac{\partial f}{\partial x_2}\right)=(\ln x_1-\ln x_2)\left(\frac{x_1-x_2}{2}\right)\geq 0.$$

According to Lemma 2.3, if $n=2$, Theorem 1.2 is true.

If $n\geq 3$, $k\geq 3$, Letting $\bar{x}=(x_3,x_4,\dots,x_n)$, $E_{n-2}(\bar{x},0)=1$, we have

$$\begin{aligned}f(x)&=P_n(x,k-1)-P_n(x,k)\\&=\left(\binom{n}{k-1}^{-1}E_n(x,k-1)\right)^{\frac{1}{k-1}}-\left(\binom{n}{k}^{-1}E_n(x,k)\right)^{\frac{1}{k}},\end{aligned}$$

$$\begin{aligned}\frac{\partial f}{\partial x_1}=&\frac{1}{k-1}\cdot\binom{n}{k-1}^{-\frac{1}{k-1}}\cdot(E_n(x,k-1))^{\frac{1}{k-1}-1}\sum_{2\leq i_1<\cdots<i_j\leq n}\prod_{j=1}^{k-2}x_{i_j}\\&-\frac{1}{k}\cdot\binom{n}{k}^{-\frac{1}{k}}\cdot(E_n(x,k))^{\frac{1}{k}-1}\sum_{2\leq i_1<\cdots<i_j\leq n}\prod_{j=1}^{k-1}x_{i_j},\end{aligned}$$

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$$x_1\frac{\partial f}{\partial x_1}=\frac{1}{k-1}\cdot\binom{n}{k-1}^{-\frac{1}{k-1}}$$
$$\cdot\left(E_n(x,k-1)\right)^{\frac{1}{k-1}-1}\left[x_1E_{n-2}(\bar{x},k-2)+x_1x_2E_{n-2}(\bar{x},k-3)\right]$$
$$-\frac{1}{k}\cdot\binom{n}{k}^{-\frac{1}{k}}\cdot\left(E_n(x,k)\right)^{\frac{1}{k}-1}\left[x_1E_{n-2}(\bar{x},k-1)+x_1x_2E_{n-2}(\bar{x},k-2)\right],$$

and

$$x_2\frac{\partial f}{\partial x_2}=\frac{1}{k-1}\cdot\binom{n}{k-1}^{-\frac{1}{k-1}}$$
$$\cdot\left(E_n(x,k-1)\right)^{\frac{1}{k-1}-1}\left[x_2E_{n-2}(\bar{x},k-2)+x_1x_2E_{n-2}(\bar{x},k-3)\right]$$
$$-\frac{1}{k}\cdot\binom{n}{k}^{-\frac{1}{k}}\cdot\left(E_n(x,k)\right)^{\frac{1}{k}-1}\left[x_2E_{n-2}(\bar{x},k-1)+x_1x_2E_{n-2}(\bar{x},k-2)\right].$$

So

$$(3.1)\quad (\ln x_1-\ln x_2)\left(x_1\frac{\partial f}{\partial x_1}-x_2\frac{\partial f}{\partial x_2}\right)$$
$$=(\ln x_1-\ln x_2)\cdot\frac{1}{k-1}\cdot\binom{n}{k-1}^{-\frac{1}{k-1}}\cdot\left(E_n(x,k-1)\right)^{\frac{1}{k-1}-1}\cdot(x_1-x_2)\cdot E_{n-2}(\bar{x},k-2)$$
$$-(\ln x_1-\ln x_2)\cdot\frac{1}{k}\cdot\binom{n}{k}^{-\frac{1}{k}}\cdot\left(E_n(x,k)\right)^{\frac{1}{k}-1}\cdot(x_1-x_2)\cdot E_{n-2}(\bar{x},k-1).$$

On the other hand, by (1.2), we deduce

$$(x_1+x_2)E_{n-2}(\bar{x},k-1)\cdot E_{n-2}(\bar{x},k-2)$$
$$+x_1x_2\left[kE_{n-2}^2(\bar{x},k-2)-(k-1)E_{n-2}(\bar{x},k-3)\cdot E_{n-2}(\bar{x},k-1)\right]\geq 0,$$

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so

$$k\cdot\left[(x_1+x_2)\,E_{n-2}(\bar{x},k-1)+x_1x_2E_{n-2}(\bar{x},k-2)\right]\cdot E_{n-2}(\bar{x},k-2)$$
$$-(k-1)\cdot\left[(x_1+x_2)\,E_{n-2}(\bar{x},k-2)+x_1x_2E_{n-2}(\bar{x},k-3)\right]\cdot E_{n-2}(\bar{x},k-1)\geq 0,$$
$$k\cdot E_n(x,k)\cdot E_{n-2}(\bar{x},k-2)-(k-1)\cdot E_n(x,k-1)\cdot E_{n-2}(\bar{x},k-1)\geq 0.$$

Again, according to (1.1), the following inequality holds.

$$k\cdot P_n(x,k-1)\cdot E_n(x,k)\cdot E_{n-2}(\bar{x},k-2)$$
$$-(k-1)\cdot P_n(x,k)\cdot E_n(x,k-1)\cdot E_{n-2}(\bar{x},k-1)\geq 0,$$

then

$$\frac{1}{k-1}\cdot\binom{n}{k-1}^{-\frac{1}{k-1}}\cdot\left(E_n(x,k-1)\right)^{\frac{1}{k-1}-1}\cdot E_{n-2}(\bar{x},k-2)$$
$$-\frac{1}{k}\cdot\binom{n}{k}^{-\frac{1}{k}}\cdot\left(E_n(x,k)\right)^{\frac{1}{k}-1}\cdot E_{n-2}(\bar{x},k-1)\geq 0.$$

Finally by (3.1), we can state that

$$(\ln x_1-\ln x_2)\left(x_1\frac{\partial f}{\partial x_1}-x_2\frac{\partial f}{\partial x_2}\right)\geq 0,$$

Thus Theorem 1.2 holds by Lemma 2.3. □

Remark 3. If $n\geq 3, k=2$, we know that f is neither a S-geometrically convex function nor a S-geometrically concave function.

Remark 4. If $n=2$, or $n\geq 3, 2\leq k-1<k\leq n$, and x logarithm majorizes y, according to Definition 2.3, we can state that

$$P_n(x,k-1)-P_n(x,k)\geq P_n(y,k-1)-P_n(y,k).$$

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By Lemma 2.2 and

$$P_n\left(\bar{G}\left(x\right), k-1\right) - P_n\left(\bar{G}\left(x\right), k\right) = 0,$$

we know that Theorem 1.2 generalizes the Maclaurin-Inequality.

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