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APPLICATION LEINDLER SPACES TO THE REAL INTERPOLATION METHOD

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ABSTRACT. The paper is devoted to the important section the Fourier analysis in one variable (AMS subject classification 42A16). In this paper we introduce Leindler space of Fourier - Haar coefficients, so we generalize [2, Theorem 7.a.12] and application to the real method spaces.

Key words and phrases: Leindler sequence space of Fourier - Haar coefficients, Lorentz space, Haar functions, real method spaces.

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1. INTRODUCTION

A Banach space $E[0,1]$ is said to be a *rearrangement invariant* space (r.i) provided $f^*(t) \leq g^*(t)$ for any $t \in [0,1]$ and $g \in E$ implies that $f \in E$ and $\|f\|_E \leq \|g\|_E$, where $g^*(t)$ is the rearrangement of $|g(t)|$. Denote by φ_E the fundamental function of (r.i) space E such that $\varphi_E = \|\kappa_e(t)\|$ (see, [1, p. 137]). Given $\tau > 0$, the dilation operator $\sigma_\tau f(t) = f(\frac{t}{\tau})$, $t \in [0,1]$ and $\min(1,\tau) \leq \|\sigma_\tau\|_{E\to E} \leq \max(1,\tau)$. Denote by

$$\alpha_E = \lim_{\tau\to+0} \frac{\ln\|\sigma_\tau\|_{E\to E}}{\ln\tau}, \qquad \beta_E = \lim_{\tau\to\infty} \frac{\ln\|\sigma_\tau\|_{E\to E}}{\ln\tau}$$

the Boyd indices of E. In general, $0 \leq \alpha_E \leq \beta_E \leq 1$.

The associated space to E' is the space of all measurable functions $f(t)$ such that $\int_0^1 f(t)g(t)dt < \infty$ for every $g(t) \in E$ endowed with the norm

$$\|f(t)\|_{E'} = \sup_{\|g(t)\|_E \leq 1} \int_0^1 f(t)g(t)dt.$$

For every (r.i) space E space the embedding $E \subset E''$ is isometric. If an (r.i) space E is separable, then (χ_n^k) is everywhere dense in E.

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Denote by Ψ the set of increasing concave functions $\psi(t) \geq 0$ on $[0,1]$ with $\psi(0)=0$. Then each function $\psi(t) \in \Psi$ generates the *Lorentz* space $\Lambda(\psi)$ endowed with the norm

$$\|g(t)\|_{\Lambda(\psi)} = \int_0^1 g^*(t) d\varphi(t) < \infty.$$

For every (r.i) space E space the embedding $E \subset E''$ is isometric.

Let be Ω the set of (n,k) such that $1 \leq k \leq 2^n$, $n \in \mathbf{N} \cup \{0\}$. Put $\chi_0^0 \equiv 1$. If $(n,k) \in \Omega$,

$$\chi_n^k(t) = \begin{cases} 1, & \frac{k-1}{2^n} < t < \frac{2k-1}{2^{n+1}}, \\ -1, & \frac{2k-1}{2^{n+1}} < t < \frac{k}{2^n}, \\ 0, & \text{for any } t \in \left[\frac{k-1}{2^n}, \ \frac{k}{2^n}\right]. \end{cases}$$

The set of functions $\left(\chi_n^k\right)$ is called the *Haar functions*, normalized in $L_\infty[0,1]$ (see [2, p. 15-18]). If an (r.i) space E is separable, then $\left(\chi_n^k\right)$ everywhere dense in E. Given $f(t) \in L_1$. The *Fourier-Haar coefficients* are given by

$$c_{n,k}(f) = 2^n \int_0^1 f(t) \chi_n^k(t) dt.$$

Put $g(t) = \sum\limits_{(n,k)\in\Omega} c_{n,k}\chi_n^k$ for any $g \in L_1[0,1]$.

A Banach sequence space E is said to be a *rearrangement invariant* space (r.i) provided that $\|(a_n)\|_E \leq \|(a_n^*)\|_E$,where a_n^* *the rearrangement of sequence* $(a_n)_{n\in\mathbf{N}}$ i.e.

$$a_n^* = \inf\left\{\sup_{i\in\mathbf{N}\setminus\mathbf{J}} |a_i| : \mathbf{J} \subset \mathbf{N}, card(\mathbf{J}) < n\right\}.$$

It is maximal if the unit ball B_E is closed in the poinwise convergence topology inducted by the space A of all real sequences. This condition is equivalent to $E^{\#} = E'$, where

$$E^{\#} = \left\{(b_n)_{n\in\mathbf{N}} \subset A : \sum_{n=1}^{\infty} |a_n b_n| < \infty, (a_n)_{n\in\mathbf{N}} \subset E\right\}$$

is the *Kother dual* of E. Clearly, $E^{\#}$ is a maximal Banach space under the norm

$$\|(b_n)\|_{E^{\#}} = \sup\left\{\sum_{n=1}^{\infty} |a_n b_n| < \infty : \|(a_n)\|_E \leq 1\right\}.$$

Denoting $\lambda = (\lambda_n)_{n=1}^\infty$ be a sequence of positive numbers. We shall use the following notation (see [3, pp. 517-518]):

$$\Lambda_n = \sum_{k=n}^{\infty} \lambda_k \text{ and } \Lambda_n^{(c)} = \sum_{k=n}^{\infty} \lambda_k \Lambda_k^{-c}, (\Lambda_1 < \infty);$$

furthermore, for $c \geq 0$. By analogy with [3, pp. 517-518] we define *Leindler sequence space of Fourier-Haar coefficients*, for $p > 0$, $c \geq 0$, with the norm:

$$\|(c_{n,k})_{n=1}^\infty\|_{\lambda(p,c)} = \left(\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left(\sum_{k=1}^{2^n} |c_{n,k}| \, 2^{-n}\right)^p\right)^{\frac{1}{p}} < \infty.$$

Why do we consider the sequence $(c_{n,k})_{n=1}^\infty$? The answer to this question follows from [2, Theorem 7.a.3], i.e. $g \in \Lambda(\psi) \Leftrightarrow \sup\limits_{0<t\leq 1} 2^{-\frac{n}{p}} c_{n,1}(g) < \infty$. Here, as usual, $X \hookrightarrow Y$ stands for the continuous embedding, that is, $\|g\|_Y \leq C\,\|g\|_X$ for some $C > 0$ and every $g \in X$. The sign $\cong$ means that these spaces coincide to with within equivalence of norms.

2. PROBLEMS

By [2, Theorem 7.a.12] for $p=2$ we have

$$\left\| \sum_{(n,k)\in\Omega} c_{n,k}\chi_n^k \right\|_{L_2} = \left(\sum_{n=1}^{\infty} 2^{-n} \sum_{k=1}^{2^n} c_{n,k}^2 \right)^{\frac{1}{2}}.$$

If for $\left\|(c_{n,k})_{n=1}^{\infty}\right\|_{\lambda(p,c)}$ we put $p=2, c=0, \lambda_n=1$, then

$$\|(c_{n,k})_{n=1}^{\infty}\|_{\lambda(2,0)} \le M \left\| \sum_{(n,k)\in\Omega} c_{n,k}\chi_n^k \right\|_{L_2}.$$

Denote by

$$T\left(\sum_{(n,k)\in\Omega} c_{n,k}\chi_n^k \right) = (c_{n,k})_{(n,k)\in\Omega}.$$

Hence by [1, Chapter 2, §5, Theorem 5.5] we have the operator bounded from $\Lambda(\psi)$ into $\lambda(2,0)$. In general we consider

Problem 1. Let $0<c<1$, $1<p<\infty$. Whether there exists a operator T bounded from $\Lambda(\psi)$ into $\lambda(p,c)$?

Let (E_0,E_1) be a compatible pair of Banach spaces. We recall

$$K(t,g) = K(t,g,E_0,E_1) = \inf_{g=g_0+g_1,\, g_i\in E_i (i=0,1)} \left(\|g_0\|_{E_0} + t\,\|g_1\|_{E_1} \right).$$

Here $g\in E_0+E_1$, $0<t\le 1$. If $0<\theta<1$, $1\le p\le\infty$, then the spaces $(E_0,E_1)_{\theta,p}$ endowed with the norm

$$\|g\|_{(E_0,E_1)_{\theta,p}} = \left(\int_0^1 (K(t,g)t^{-\theta})^p \frac{dt}{t} \right)^{\frac{1}{p}} < \infty, \text{ iff } p<\infty$$

and

$$\|g\|_{(E_0,E_1)_{\theta,p}} = \sup_{0<t<1} K(t,g)t^{-\theta} < \infty, \text{ iff } p=\infty$$

are called real method spaces. Let $0\le\alpha_0<\alpha_1<1$, $\psi_0(t)=t^{\alpha_0}$, $\psi_{1,}(t)=t^{\alpha_1}$, $0<\theta<1$, $1\le p\le\infty$, $\widetilde{\psi}(t)=\frac{t}{\psi(t)}$. In [5, §2, p. 174] the problem was solved: when does the equivalence

$$(\Lambda(\psi_0),\Lambda(\psi_1))_{\theta,p} \cong \left(M(\widetilde{\psi_0}), M(\widetilde{\psi_1}) \right)_{\theta,p}.$$

holds?

We consider the embedding $(\Lambda(\psi_0),\Lambda(\psi_1))_{\theta,p} \hookrightarrow \left(M(\widetilde{\psi_0}), M(\widetilde{\psi_1}) \right)_{\theta,p}$. Let $0\le\alpha_0=\alpha_1<1$, $\psi(t)=t^{\alpha}$, $0<\theta<1$, $1<p\le\infty$.

Problem 2. Whether there exists $0<c<1, 1<p<\infty$ such that

$$T : (\Lambda(\psi),\Lambda(\psi))_{\theta,p} \to (\lambda(p,c),\lambda(p,c))_{\theta,p}\ ?$$

In this article we consider Leindler sequence space of Fourier-Haar coefficients $\lambda(p,c)$.

To prove our theorems we need the following Theorem 1 (see [4]).

Theorem 1. *If $p > 1, 0 \leq c < 1$, then*

$$\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left(\sum_{k=1}^{n} |a_k| \right)^p \leq \left(\frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda_n^{1-p} \Lambda_n^{p-c} a_n^p .$$

The constant is best possible.

3. LEMMAS AND THEOREMS

Lemma 3.1. *Let $1 < p < \infty, 0 \leq c < 1$ and $\sup_{0<t\leq 1} 2^{-\frac{n}{p}} c_{n,1}(g) < \infty$. Then the operator T is bounded from $\Lambda(\psi)$ into $\lambda(p,c)$.*

Proof. By [2, Theorem 4.a.1] for $1 < p < \infty$ we have

$$\int_0^1 \left| \sum_{k=1}^{2^n} c_{n,k} \chi_n^k \right|^p dt \leq \int_0^1 \left| \sum_{n=l}^{\infty} \sum_{k=1}^{2^n} c_{n,k} \chi_n^k \right|^p dt \leq 2^p \int_0^1 \left| \sum_{(n,k)\in\Omega} c_{n,k} \chi_n^k \right|^p dt,$$

where $n \leq l \leq \infty$.

On the other hand,

$$\int_0^1 \left\| \sum_{k=1}^{2^n} c_{n,k} \chi_n^k \right\|_{L_p}^p dt = \int_0^1 2^{-n} \sum_{k=1}^{2^n} |c_{n,k}|^p \, dt = 2^{-n} \sum_{k=1}^{2^n} |c_{n,k}|^p .$$

Therefore,

$$\left(2^{-n} \sum_{k=1}^{2^n} |c_{n,k}|^p \right)^{\frac{1}{p}} \leq 2 \, \|g\|_{L_p} .$$

From the above and [1, Chapter 2, §5, Theorem 5.5] we get

$$\|(c_{n,k})_{n=1}^{\infty}\|_{\lambda(p,c)} \leq 2 \left(\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \right)^{\frac{1}{p}} \|g\|_{\Lambda(\psi)} .$$

Hence the operator T is bounded from $\Lambda(\psi)$ into $\lambda(p,c)$. This proves the assertion. □

Remark 3.2. In the Lemma 3.1 the condition $0 < c < 1, 1 < p < \infty$ is necessary for the operator T.

We shall formulate the sufficient condition of boundedness of the operator T from $\Lambda(\psi)$ into $\lambda(p,c)$.

Theorem 3.3. *Let $0 \leq c < 1$, $\sup_{0<t\leq 1} 2^{-\frac{n}{p}} c_{n,1}(g) < \infty$. For of boundedness the operator T bounded from $\Lambda(\psi)$ into $\lambda(p,c)$ is sufficient that $2 \leq p < \infty$.*

Proof. By Theorem 1 and Hölder's inequality we have

$$\|(c_{n,k})_{n=1}^{\infty}\|_{\lambda(p,c)} \leq \frac{p}{1-c} \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{p}-1} \Lambda_n^{1-\frac{c}{p}} \left(\sum_{(n,k)\in\Omega} |c_{n,k}|^p \, 2^{-n} \right)^{\frac{1}{p}} .$$

Now using [2, Theorem 7.a.12 (c. 2)] and [1, Chapter 2, §5, Theorem 5.5] we obtain that

$$\left\|(c_{n,k})_{n=1}^{\infty}\right\|_{\lambda(p,c)} \leq \frac{p}{1-c} \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{p}-1} \Lambda_n^{1-\frac{c}{p}} \left\| \sum_{(n,k)\in\Omega} c_{n,k} \chi_n^k \right\|_{\Lambda(\psi)} .$$

This finishes the proof. □

Remark 3.4. If $1 \leq p < 2$, $0 < c < 1$, then by [2, Theorem 7.a.12 (c. 1)] $T : \Lambda(\psi) \nrightarrow \lambda(p,c)$.

Theorem 3.5. *Let* $0 \leq c < 1, 2 \leq p \leq \infty, \sup\limits_{0<t\leq 1} 2^{-\frac{n}{p}} c_{n,1}(g) < \infty$. *Then*

$$T : (\Lambda(\psi), \Lambda(\psi))_{\theta,p} \to (\lambda(p,c), \lambda(p,c))_{\theta,p}.$$

Proof. Clearly, by Hölder's inequality the estimate

$$\left\|(c_{n,k})_{n=1}^{\infty}\right\|_{\lambda(p,c)} \leq \frac{p}{1-c} \sum_{n=1}^{\infty} \lambda_n^{\frac{1}{p}-1} \Lambda_n^{1-\frac{c}{p}} \left\|(c_{n,k})_{n=1}^{\infty}\right\|_{\ell_2}$$

holds. It is known that the operator T is bounded from L_2 into ℓ_2. Then from the above and [1, Chapter 2, §5, Theorem 5.5] we obtain

$$K(t, (c_{n,k})_{n=1}^{\infty}, \lambda(p,c), \lambda(p,c)) \leq K(t, g, \Lambda(\psi), \Lambda(\psi)).$$

Hence $T : (\Lambda(\psi), \Lambda(\psi))_{\theta,p} \to (\lambda(p,c), \lambda(p,c))_{\theta,p}$. This completes the proof. □

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