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ASYMPTOTIC EXPANSION OF THE EQUIPOISE CURVE OF A POLYNOMIAL INEQUALITY

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ABSTRACT. For any $\mathbf{a} := (a_1, a_2, \dots, a_n) \in (\mathbb{R}^+)^n$, define $\Delta P_{\mathbf{a}}(x,t) := (x + a_1 t)(x + a_2 t) \cdots (x + a_n t) - x^n$ and $S_{\mathbf{a}}(x,y) := a_1 x^{n-1} + a_2 x^{n-2} y + \cdots + a_n y^{n-1}$. The two homogeneous polynomials $\Delta P_{\mathbf{a}}(x,t)$ and $tS_{\mathbf{a}}(x,y)$ are comparable in the positive octant $x, y, t \in \mathbb{R}^+$. Recently the authors [2] studied the inequality $\Delta P_{\mathbf{a}}(x,t) > tS_{\mathbf{a}}(x,y)$ and its reverse and noted that the boundary between the corresponding regions in the positive octant is fully determined by the equipoise curve $\Delta P_{\mathbf{a}}(x,1) = S_{\mathbf{a}}(x,y)$. In the present paper the asymptotic expansion of the equipoise curve is shown to exist, and is determined both recursively and explicitly. Several special cases are then examined in detail, including the general solution when $n = 3$, where the coefficients involve a type of generalised Catalan number, and the case where $\mathbf{a} = \mathbf{1} + \delta$ is a sequence in which each term is close to 1. A selection of inequalities implied by these results completes the paper.

Key words and phrases: Polynomial inequality, Catalan numbers.

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1. INTRODUCTION

With any finite real sequence $\mathbf{a} := (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ we associate two homogeneous polynomials, the *product* polynomial

$$P_{\mathbf{a}}(x,t) := (x + a_1 t)(x + a_2 t) \cdots (x + a_n t) = \prod_{r=1}^{n} (x + a_r t),$$

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and the *sum* polynomial

$$S_{\mathbf{a}}(x,y) := a_1x^{n-1} + a_2x^{n-2}y + \cdots + a_ny^{n-1} = \sum_{r=1}^{n} a_rx^{n-r}y^{r-1}.$$

As shown in [2], the *first difference* of the product polynomial

$$\Delta P_{\mathbf{a}}(x,t) := P_{\mathbf{a}}(x,t) - P_{\mathbf{a}}(x,0) = P_{\mathbf{a}}(x,t) - x^n$$

and t times the sum polynomial are degree n homogeneous polynomials which are comparable in the positive octant $x, y, t \in \mathbb{R}^+ := \{r \in \mathbb{R} : r \geq 0\}$ when $\mathbf{a} \in (\mathbb{R}^+)^n$. Clearly they are closely related to the comparison of the product $\Pi_{r=1}^n(1+a_r)$ and the sum $\Sigma_{r=1}^n a_r$. Indeed Weierstrass [4] derived inequalities equivalent to

$$1 + \sum_{r=1}^{n} a_r < \prod_{r=1}^{n}(1+a_r) < \frac{1}{\prod_{r=1}^{n}(1-a_r)} < \frac{1}{1-\sum_{r=1}^{n} a_r}$$

when $\mathbf{a} \in (\mathbb{R}^+)^n$ and $0 < \Sigma_{r=1}^n a_r < 1$, whence the products $\Pi_{r=1}^n(1+a_r)$ and $\Pi_{r=1}^n(1-a_r)$ both converge as $n \to \infty$ if $\Sigma_{r=1}^n a_r$ converges to a limit strictly less than 1. The first and third of these inequalities correspond to $S_{\mathbf{a}}(1,1) < \Delta P_{\mathbf{a}}(1,1)$ and $-S_{\mathbf{a}}(1,1) < \Delta P_{\mathbf{a}}(1,-1)$ respectively, while the middle inequality simply follows from $1 - a_r^2 \leq 1$ for $1 \leq r \leq n$, with strict inequality for at least one r. The first inequality corresponds to results in [2] at the point $(x,y,t) = (1,1,1)$, but the third corresponds to $(1,1,-1)$, which is outside the positive octant, and, although easily proved, it is not covered in [2]. (A more widely accessible source which closely parallels Weierstrass's reasoning is [1].)

To summarise the results in [2], let us now suppose that $n \geq 2$ and $\mathbf{a}$ is strictly positive, so $a_r > 0$ for $1 \leq r \leq n$. Then the strict inequality $\Delta P_{\mathbf{a}}(x,t) > tS_{\mathbf{a}}(x,y)$ holds in a region (the "ΔP-region") of the positive octant which includes the intersection of the octant with the halfspace $y < x + tm(\mathbf{a})$, where $m(\mathbf{a}) := \min\{a_r : 1 \leq r \leq n-1\}$, and the reverse inequality $\Delta P_{\mathbf{a}}(x,t) < tS_{\mathbf{a}}(x,y)$ holds in a region (the "S-region") of the positive octant which includes its intersection with the halfspace $y > x + tM(\mathbf{a})$, where $M(\mathbf{a}) := \max\{a_r : 1 \leq r \leq n-1\}$. The boundary between the ΔP-region and the S-region is the *equipoise surface*

$$E_2(\mathbf{a}) := \{(x,y,t) \in (\mathbb{R}^+)^3 : \Delta P_{\mathbf{a}}(x,t) = tS_{\mathbf{a}}(x,y)\}.$$

The polynomials are homogeneous in $\mathbf{a}$, so for any real t we have

$$\Delta P_{t\mathbf{a}}(x,1) = \Delta P_{\mathbf{a}}(x,t) \quad \text{and} \quad S_{t\mathbf{a}}(x,y) = tS_{\mathbf{a}}(x,y),$$

where $t\mathbf{a} := (ta_1, ta_2, \ldots, ta_n) \in \mathbb{R}^n$. Hence for strictly positive $\mathbf{a} \in (\mathbb{R}^+)^n$ with $n \geq 2$, it suffices to compare the polynomials in the intersection of the positive octant with the plane $t = 1$, so we consider the *equipoise curve*

$$E_1(\mathbf{a}) := \{(x,y) \in (\mathbb{R}^+)^2 : \Delta P_{\mathbf{a}}(x,1) = S_{\mathbf{a}}(x,y)\}.$$

This separates the ΔP-region of the positive quadrant $x, y \in \mathbb{R}^+$, where $\Delta P_{\mathbf{a}}(x,1) > S_{\mathbf{a}}(x,y)$, from the S-region, where $\Delta P_{\mathbf{a}}(x,1) < S_{\mathbf{a}}(x,y)$. The equipoise curve lies in the strip

$$x + m(\mathbf{a}) \leq y \leq x + M(\mathbf{a})$$

of the positive quadrant and is asymptotic to $y = x + \alpha$, where α is a certain function of $\mathbf{a}$. In fact, if $n \geq 3$ the equipoise curve satisfies

$$y = x + \alpha + \beta x^{-1} + O(x^{-2}) \qquad \text{as } x \to \infty$$

where α, β are functions of $\mathbf{a}$ explicitly determined in [2]. The equipoise curve approaches the asymptote from the ΔP-region side if β is negative, and from the S-region side if β is positive.

Our main purpose in this paper is to extend our understanding of α and β as functions of $\mathbf{a}$, by determining the subsequent members of an infinite sequence of coefficients constituting the asymptotic expansion of the equipoise curve for $\mathbf{a}$. But first we shall show that the properties just summarized hold a little more generally.

2. WIDER RANGE OF VALIDITY

To extend the results of [2] it is convenient to introduce some notation. For any sequence $\mathbf{a} := (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$ and any integer k in the interval $0 \le k \le n$, let

$$A_k(\mathbf{a}) := (a_1, a_2, \ldots, a_k) \quad \text{and} \quad \Omega_k(\mathbf{a}) := (a_{n-k+1}, \ldots, a_{n-1}, a_n)$$

be, respectively, the *initial* and *final* k-term subsequences of $\mathbf{a}$. Thus $A_n(\mathbf{a}) = \Omega_n(\mathbf{a}) = \mathbf{a}$ and, if ω is the empty sequence, $A_0(\mathbf{a}) = \Omega_0(\mathbf{a}) = \omega$. Also $m(\mathbf{a}) := \min\{a_r : 1 \le r \le n-1\} = \min A_{n-1}(\mathbf{a})$ and $M(\mathbf{a}) := \max\{a_r : 1 \le r \le n-1\} = \max A_{n-1}(\mathbf{a})$. As in [2] we also use $\Sigma(\mathbf{a}) := \Sigma_{r=1}^n a_r$.

First, a simple reformulation of Corollary 2.2 of [2] becomes

Theorem 2.1. *For any finite sequence* $\mathbf{a} \in (\mathbb{R}^+)^n$ *with* $n \ge 3$, *and for all strictly positive* $x, y, t \in \mathbb{R}^+$, *if* $A_{n-1}(\mathbf{a})$ *is not constant then for* $y \ge x + t \max A_{n-1}(\mathbf{a})$ *we have*

$$0 < t\Sigma(\mathbf{a})x^{n-1} < \Delta P_{\mathbf{a}}(x,t) < tS_{\mathbf{a}}(x,y),$$

while for $y \le x + t \min A_{n-1}(\mathbf{a})$ *and* $z := \min\{x, y\}$ *we have*

$$\Delta P_{\mathbf{a}}(x,t) > tS_{\mathbf{a}}(x,y) \ge t\Sigma(\mathbf{a})z^{n-1} > 0.$$

To investigate the equality $\Delta P_{\mathbf{a}}(x,1) = S_{\mathbf{a}}(x,y)$, in [2] we imposed the sufficient condition that $\mathbf{a} \in (\mathbb{R}^+)^n$ be strictly positive. However, we note that if x, y are strictly positive then

$$\frac{\partial}{\partial y} S_{\mathbf{a}}(x,y) > 0$$

holds if and only if $\Omega_{n-1}(\mathbf{a}) \ne \mathbf{0}$, where $\mathbf{0} \in \mathbb{R}^{n-1}$ is the constant sequence with every term equal to 0. We shall abbreviate this condition by saying "if and only if $\Omega_{n-1}(\mathbf{a})$ is nonzero". Then continuity of $S_{\mathbf{a}}(x,y)$ as a function of y ensures the following broadening of the scope of Lemma 3.1 of [2]:

Theorem 2.2. *For any finite sequence* $\mathbf{a} \in (\mathbb{R}^+)^n$ *with* $n \ge 2$, *and strictly positive* $x, y \in \mathbb{R}^+$, *if* $\Omega_{n-1}(\mathbf{a})$ *is nonzero then there is a function* $y_0(x)$ *such that*

$$\Delta P_{\mathbf{a}}(x,1) \begin{cases} < & S_{\mathbf{a}}(x,y) \quad \text{if } y > y_0(x), \\ = & S_{\mathbf{a}}(x,y) \quad \text{if } y = y_0(x), \\ > & S_{\mathbf{a}}(x,y) \quad \text{if } y < y_0(x). \end{cases}$$

Furthermore

$$x + \min A_{n-1}(\mathbf{a}) \le y_0(x) \le x + \max A_{n-1}(\mathbf{a}).$$

As in [2], it is convenient now to define two families of *sequence functions* $\Sigma_k, W_k : \mathbb{R}^n \to \mathbb{R}$, for any positive integer n and all positive integers $k \le n$. These functions are needed to describe the coefficients in the asymptotic expansion of $y = y_0(x)$, the equipoise curve for $\mathbf{a}$.

The kth *elementary symmetric function* Σ_k of $\mathbf{a} \in \mathbb{R}^n$ is the sum of all products $\Pi\mathbf{x}$ as $\mathbf{x}$ runs through the k-term subsequences $\mathbf{x} \subseteq \mathbf{a}$, thus

$$\Sigma_k(\mathbf{a}) := \Sigma\left\{\Pi\mathbf{x} : \mathbf{x} \subseteq \mathbf{a}, |\mathbf{x}| = k\right\}.$$

In particular, $\Sigma_1(\mathbf{a}) = \Sigma_{r=1}^n a_r$ and $\Sigma_2(\mathbf{a}) = \Sigma_{r=1}^{n-1}\Sigma_{s=r+1}^n a_r a_s$ if $n \ge 2$. We extend the definition by setting $\Sigma_k(\mathbf{a}) = 0$ for any integer $k > n$.

The kth *binomially-weighted sum* W_k of $\mathbf{a} \in \mathbb{R}^n$ is the sequence function

$$W_k(\mathbf{a}) := \sum_{r=1}^{n} \binom{r-1}{k-1} a_r.$$

In particular, $W_1(\mathbf{a}) = \Sigma_{r=1}^n a_r$ and $W_2(\mathbf{a}) = \Sigma_{r=1}^n (r-1)a_r$ if $n \geq 2$. Note that $W_1(\mathbf{a}) = \Sigma_1(\mathbf{a})$ holds for any $\mathbf{a}$. Once again we extend the definition by setting $W_k(\mathbf{a}) = 0$ for any integer $k > n$. Now Theorem 2.2 justifies the following broadening of the scope of Theorem 3.1 and Corollary 3.2 of [2].

Theorem 2.3. *For any finite sequence $\mathbf{a} \in (\mathbb{R}^+)^n$ with $n \geq 2$, and strictly positive $x, y \in \mathbb{R}^+$, if $\Omega_{n-1}(\mathbf{a})$ is nonzero then the equality $\Delta P_{\mathbf{a}}(x, 1) = S_{\mathbf{a}}(x, y)$ holds for large x when*

$$y = x + \alpha + \beta x^{-1} + O(x^{-2}) \quad \text{as } x \to \infty,$$

where

$$\alpha := \Sigma_2(\mathbf{a})/W_2(\mathbf{a}) \quad \text{and} \quad \beta := (\Sigma_3(\mathbf{a}) - \alpha^2 W_3(\mathbf{a}))/W_2(\mathbf{a}).$$

Note that $\beta = 0$ if $n = 2$. We will extend Theorem 2.3 in the next section.

3. Asymptotic Expansion of the Equipoise Curve

Let us first establish the existence of the asymptotic expansion of $E_1(\mathbf{a})$ for suitable $\mathbf{a}$.

Theorem 3.1. *For any finite sequence $\mathbf{a} \in (\mathbb{R}^+)^n$ with $n \geq 2$ and $\Omega_{n-1}(\mathbf{a})$ nonzero, there is an infinite sequence $\alpha := (\alpha_1, \alpha_2, \ldots) \in \mathbb{R}^\infty$ such that the equipoise curve $E_1(\mathbf{a})$ has asymptotic expansion*

$$y \sim x\left(1 + \sum_{s=1}^{\infty} \alpha_s x^{-s}\right) \quad \text{as } x \to \infty.$$

Proof. By Theorem 2.3, there is an $\alpha_1 \in \mathbb{R}$ such that $E_1(\mathbf{a})$ is $y = x + \alpha_1 + O(x^{-1})$ as $x \to \infty$. Now assume for some positive integer N that there is a sequence $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in \mathbb{R}^N$ such that $E_1(\mathbf{a})$ is

$$y = x\left(1 + \sum_{s=1}^{N} \alpha_s x^{-s}\right) + f_N(x),$$

with $O(f_N(x)) = O(x^{-N})$ as $x \to \infty$. Then

$$\begin{aligned}
\Delta P_{\mathbf{a}}(x, 1) &= S_{\mathbf{a}}(x, y) \\
&= \sum_{r=1}^{n} a_r x^{n-r} \left(x\left(1 + \sum_{s=1}^{N} \alpha_s x^{-s}\right) + f_N(x) \right)^{r-1} \\
&= \sum_{r=1}^{n} a_r x^{n-1} \left(1 + \sum_{s=1}^{N} \alpha_s x^{-s}\right)^{r-1} + \sum_{r=1}^{n} (r-1) a_r x^{n-2} f_N(x) + O(x^{n-N-3}).
\end{aligned}$$

Note that $O(x^{n-2} f_N(x)) = O(x^{n-N-2})$. Our assumption for N implies that coefficients of powers of x down as far as x^{n-N-1} on the right match the corresponding coefficients in $\Delta P_{\mathbf{a}}(x, 1)$, so it follows that

$$\sum_{r=1}^{n} (r-1) a_r f_N(x) = c x^{-N} + O(x^{-N-1}),$$

where the coefficient c is equal to the difference between the coefficients of x^{n-N-2} in $\Delta P_{\mathbf{a}}(x,1)$ and in $\Sigma_{r=1}^n a_r x^{n-1}(1+\Sigma_{s=1}^N \alpha_s x^{-s})^{r-1}$. The coefficient of $f_N(x)$ is nonzero because $\Omega_{n-1}(\mathbf{a})$ is nonzero. Let $\alpha_{N+1} := c/\Sigma_{r=1}^n (r-1)a_r$. Then $E_1(\mathbf{a})$ is

$$y = x\left(1+\sum_{s=1}^{N+1}\alpha_s x^{-s}\right) + O(x^{-N-1}).$$

The theorem now follows by induction on N. $\square$

Under the conditions of Theorem 3.1, the equipoise curve $E_1(\mathbf{a})$ has an asymptotic expansion with coefficient sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ as $x \to \infty$. Since $W_2(\mathbf{a}) = \Sigma_{r=1}^n (r-1)a_r$, the proof of Theorem 3.1 shows that $\alpha_N = c_N/W_2(\mathbf{a})$, where c_N is the difference between the coefficients of x^{-N} in the expansions

$$\frac{\Delta P_{\mathbf{a}}(x,1)}{x^{n-1}} = \Sigma_1(\mathbf{a}) + \Sigma_2(\mathbf{a})x^{-1} + \Sigma_3(\mathbf{a})x^{-2} + \ldots = \sum_{k=0}^{\infty} \Sigma_{k+1}(\mathbf{a})x^{-k}$$

and

$$\sum_{r=1}^n a_r \left(1+\sum_{s=1}^{N-1}\alpha_s x^{-s}\right)^{r-1} := \sum_{k=0}^{\infty} C_{N,k}(\mathbf{a})x^{-k},$$

so $c_N = \Sigma_{N+1}(\mathbf{a}) - C_{N,N}(\mathbf{a})$. Of course, we have yet to determine the coefficients $C_{N,k}(\mathbf{a})$, but note immediately that $C_{N,k}(\mathbf{a}) = 0$ for all sufficiently large k.

Let $\mathbf{d} := (d_1, \ldots, d_{N-1}) \in (\mathbb{Z}^+)^{N-1}$ be a nonnegative integer sequence such that $\Sigma_{s=1}^{N-1} s d_s = k$ and $\Sigma_{s=1}^{N-1} d_s = m$. Then $\mathbf{d}$ is a *partition* of k with *length* $N-1$ and *weight* m. Corresponding to each $\mathbf{d}$ with weight $m \le r-1$, there is a term in x^{-k} in the expansion of $\left(1+\Sigma_{s=1}^{N-1}\alpha_s x^{-s}\right)^{r-1}$, with coefficient

$$\frac{(r-1)!}{(r-m-1)!d_1!d_2!\ldots d_{N-1}!} \cdot \alpha_1^{d_1}\alpha_2^{d_2}\ldots\alpha_{N-1}^{d_{N-1}}.$$

For convenience we abbreviate such expressions with the following compact notation for the product

$$\alpha(\mathbf{d}) := \prod_{s=1}^{N-1}\alpha_s^{d_s}$$

and the multinomial coefficient

$$\binom{\Sigma(\mathbf{d})}{\mathbf{d}} := \frac{m!}{\Pi_{s=1}^{N-1} d_s!},$$

where $\Sigma(\mathbf{d}) = m$. Thus the coefficient of the term in x^{-k} corresponding to $\mathbf{d}$ in the expansion of $\left(1+\Sigma_{s=1}^{N-1}\alpha_s x^{-s}\right)^{r-1}$ becomes

$$\binom{r-1}{\Sigma(\mathbf{d})}\binom{\Sigma(\mathbf{d})}{\mathbf{d}}\alpha(\mathbf{d}),$$

where the first factor is the binomial coefficient $\binom{r-1}{m}$, which by definition is 0 when $m > r-1$.

Let $P(k, N-1, m) \subseteq (\mathbb{Z}^+)^{N-1}$ be the set of all partitions of k with length $N-1$ and weight m. Then in the expansion of $\Sigma_{r=1}^n a_r \left(1+\Sigma_{s=1}^{N-1}\alpha_s x^{-s}\right)^{r-1}$ the coefficient of the term in x^{-k} corresponding to any particular $\mathbf{d} \in P(k, N-1, m)$ is

$$\sum_{r=1}^n \binom{r-1}{m} a_r \binom{m}{\mathbf{d}}\alpha(\mathbf{d}) = \binom{m}{\mathbf{d}}\alpha(\mathbf{d})W_{m+1}(\mathbf{a}).$$

Summing over all partitions in $P(k, N-1, m)$ and all relevant weights m yields $C_{N,k}(\mathbf{a})$, the total coefficient of x^{-k}. When $k = N$ we obtain the coefficient $C_{N,N}(\mathbf{a})$ needed for α_N. Simplifying notation with $P(N, m) := P(N, N-1, m)$, and noting that $P(N, 1) = \emptyset$, we have

Theorem 3.2. *For any finite sequence $\mathbf{a} \in (\mathbb{R}^+)^n$ with $n \geq 2$ and $\Omega_{n-1}(\mathbf{a})$ nonzero, the asymptotic expansion of the equipoise curve $E_1(\mathbf{a})$ is*

$$y \sim x\left(1 + \sum_{s=1}^{\infty} \alpha_s x^{-s}\right) \quad \textit{as } x \to \infty,$$

where the coefficient sequence $\alpha := (\alpha_1, \alpha_2, \dots) \in \mathbb{R}^\infty$ is given by

$$\alpha_N = \frac{\Sigma_{N+1}(\mathbf{a}) - C_{N,N}(\mathbf{a})}{W_2(\mathbf{a})}$$

for each $N \geq 1$, with

$$C_{N,N}(\mathbf{a}) = \sum_{m=2}^{N}\left(\sum_{\mathbf{d} \in P(N,m)} \binom{m}{\mathbf{d}} \alpha(\mathbf{d})\right) W_{m+1}(\mathbf{a}).$$

In particular, when $N = 1$ we have $P(1, m) = \emptyset$ so $C_{1,1}(\mathbf{a}) = 0$, since its inner sum is empty. This gives $\alpha_1 = \Sigma_2(\mathbf{a}) / W_2(\mathbf{a})$, consistent with Theorem 2.3. Again, when $N = 2$ the sequence $(2) \in \mathbb{R}^1$ is the unique partition of 2 with length 1, so $P(2, 2) = \{(2)\}$ and the inner sum for $C_{2,2}(\mathbf{a})$ is $\binom{2}{2}\alpha(2) = \alpha_1^2$, whence $C_{2,2}(\mathbf{a}) = \alpha_1^2 W_3(\mathbf{a})$. Then $\alpha_2 = (\Sigma_3(\mathbf{a}) - \alpha_1^2 W_3(\mathbf{a})) / W_2(\mathbf{a})$, again consistent with Theorem 2.3. Substituting here for α_1 and suppressing the argument $\mathbf{a}$ to simplify notation yields

$$\alpha_2 = \frac{\Sigma_3 W_2^2 - \Sigma_2^2 W_3}{W_2^3}.$$

When $N = 3$ we have $P(3, 2) = \{(1, 1)\}$ and $P(3, 3) = \{(3, 0)\}$, so Theorem 3.2 yields α_3 from $C_{3,3}(\mathbf{a}) = 2\alpha_1\alpha_2 W_3(\mathbf{a}) + \alpha_1^3 W_4(\mathbf{a})$. Substituting for α_1 and α_2 and suppressing the argument $\mathbf{a}$ now yields

$$\alpha_3 = \frac{\Sigma_4 W_2^4 - \Sigma_2^3 W_2 W_4 + 2\Sigma_2^3 W_3^2 - 2\Sigma_2\Sigma_3 W_2^2 W_3}{W_2^5}.$$

Evidently continuing this process will yield an expression for any α_N just in terms of the elementary symmetric functions and the binomially-weighted functions of the sequence $\mathbf{a}$. In the next theorem we characterize the summands in this explicit expression for α_N, but first we introduce some notation. For any integer sequence $\mathbf{d} \in (\mathbb{Z}^+)^N$ let

$$\boldsymbol{\Sigma}(\mathbf{d}) := \prod_{r=1}^{N} \Sigma_{r+1}(\mathbf{a})^{d_r} \quad \text{and} \quad \mathbf{W}(\mathbf{d}) := \prod_{r=1}^{N} W_{r+1}(\mathbf{a})^{d_r}.$$

As previously, we shall usually suppress explicit mention of the argument $\mathbf{a}$ from expressions of this type. We also need the following family of partition pairs:

$$Q(N) := \left\{(\mathbf{d}, \mathbf{e}) : \mathbf{d}, \mathbf{e} \in (\mathbb{Z}^+)^N, \sum_{r=1}^{N} r d_r = N, \sum_{r=1}^{N} r e_r = 2N-2, \sum_{r=1}^{N}(d_r + e_r) = 2N-1\right\},$$

that is, pairs $(\mathbf{d}, \mathbf{e})$ of partitions of N and $2N-2$ respectively, each of length N, with sum of weights equal to $2N-1$. (This places no effective restriction on $\mathbf{d}$, but does constrain $\mathbf{e}$ significantly.)

Theorem 3.3. *For each integer $N \geq 1$, the coefficient α_N in the asymptotic expansion of the equipoise curve $E_1(\mathbf{a})$ satisfies an identity of the form*

$$\alpha_N W_2^{2N-1} = \sum_{(\mathbf{d},\mathbf{e}) \in Q(N)} c(\mathbf{d}, \mathbf{e}) \mathbf{\Sigma}(\mathbf{d}) \mathbf{W}(\mathbf{e}),$$

where each coefficient $c(\mathbf{d}, \mathbf{e})$ is an integer dependent only on the partition pair $(\mathbf{d}, \mathbf{e})$.

Proof. Note that $Q(1) = \{((1), (0))\}$ and $\alpha_1 W_2 = \Sigma_2$, so the theorem holds when $N = 1$, with $c((1), (0)) = 1$. Now fix $N > 1$, and suppose inductively that the theorem holds for all α_s with $1 \leq s < N$. By Theorem 3.2,

$$\alpha_N W_2^{2N-1} = \Sigma_{N+1} W_2^{2N-2} - \sum_{M=2}^{N} \left(\sum_{\mathbf{D} \in P(N,M)} \binom{M}{\mathbf{D}} \alpha(\mathbf{D}) \right) W_2^{2N-2} W_{M+1}.$$

The first term on the right is of the required form, since it is $c(\mathbf{d}, \mathbf{e}) \mathbf{\Sigma}(\mathbf{d}) \mathbf{W}(\mathbf{e})$ with $c(\mathbf{d}, \mathbf{e}) = 1$, where $\mathbf{d} = (0, \ldots, 0, 1), \mathbf{e} = (2N-2, 0, \ldots, 0) \in (\mathbb{Z}^+)^N$ are length N partitions of N and $2N-2$ respectively, with sum of weights $2N-1$.

Now consider the outer sum on the right in the α_N identity. Each summand is of the form

$$\left(\sum_{\mathbf{D} \in P(N,M)} \binom{M}{\mathbf{D}} W_2^{2N-M} \alpha(\mathbf{D}) \right) W_2^{M-2} W_{M+1}.$$

Since $2N - M = \Sigma_{s=1}^{N-1} (2s-1) D_s$ for each $\mathbf{D} \in P(N, M)$, we have

$$W_2^{2N-M} \alpha(\mathbf{D}) = \prod_{s=1}^{N-1} (\alpha_s W_2^{2s-1})^{D_s} = \prod_{s=1}^{N-1} \left(\sum_{(\mathbf{d},\mathbf{e}) \in Q(s)} c(\mathbf{d}, \mathbf{e}) \mathbf{\Sigma}(\mathbf{d}) W(\mathbf{e}) \right)^{D_s}$$

where the last step is by hypothesis. If $(\mathbf{d}, \mathbf{e}), (\mathbf{d}', \mathbf{e}') \in Q(s)$ then

$$\mathbf{\Sigma}(\mathbf{d}) \mathbf{W}(\mathbf{e}) \cdot \mathbf{\Sigma}(\mathbf{d}') \mathbf{W}(\mathbf{e}') = \mathbf{\Sigma}(\mathbf{d} + \mathbf{d}') \mathbf{W}(\mathbf{e} + \mathbf{e}'),$$

so it follows that every term in the expansion of the sum over $Q(s)$, raised to the power D_s, is of the form $c(\mathbf{d}, \mathbf{e}) \mathbf{\Sigma}(\mathbf{d}) \mathbf{W}(\mathbf{e})$ where $c(\mathbf{d}, \mathbf{e})$ is an integer and $\mathbf{d}, \mathbf{e} \in (\mathbb{Z}^+)^s$ are partitions of sD_s and $(2s-2)D_s$ respectively, with sum of weights $(2s-1)D_s$. To calculate the product over s, we modify these length s partitions by adjoining a further $N - s$ zero terms to each. Partitions corresponding to different values of s can then be added. The sum of $N-1$ pairs, one for each value of s, is a pair $(\mathbf{d}, \mathbf{e})$ of length N partitions, where $\mathbf{d}$ is a partition of $\Sigma_{s=1}^{N-1} s D_s = N$ and $\mathbf{e}$ is a partition of $\Sigma_{s=1}^{N-1} (2s-2) D_s = 2N - 2M$, and the sum of weights of $\mathbf{d}$ and $\mathbf{e}$ is $\Sigma_{s=1}^{N-1} (2s-1) D_s = 2N - M$. Before we sum over M, recall that each such term is multiplied by $W_2^{M-2} W_{M+1} = W(\mathbf{e}^*)$, where $\mathbf{e}^* \in (\mathbb{Z}^+)^N$ has $e_1^* = M-2, e_M^* = 1$ and all other terms 0. Thus $\mathbf{e}^*$ is a length N partition of $2M-2$ with weight $M-1$. Hence $(\mathbf{d}, \mathbf{e} + \mathbf{e}^*)$ is a pair of length N partitions of N and $2N-2$ respectively, with sum of weights $2N-1$, so $(\mathbf{d}, \mathbf{e} + \mathbf{e}^*) \in Q(N)$. This does not depend explicitly on M, so the sum over M is a sum of terms of the form $c(\mathbf{d}, \mathbf{e}) \mathbf{\Sigma}(\mathbf{d}) \mathbf{W}(\mathbf{e})$ where $(\mathbf{d}, \mathbf{e}) \in Q(N)$. All coefficients $c(\mathbf{d}, \mathbf{e})$ involve sums of products of multinomial coefficients and integer coefficients from α_s with $1 \leq s < N$, so every $c(\mathbf{d}, \mathbf{e})$ is an integer. The theorem now follows by induction on N. $\square$

For example, the set $Q(4)$ comprises eight partition pairs, each pair being a length 4 partition of 4 and a length 4 partition of 6, with sum of weights 7. Thus $\alpha_4 W_2^7$ is a sum of eight products of Σ_k's and W_k's, with coefficients as noted:

$(\mathbf{d},\mathbf{e}) \in Q(4)$	$c(\mathbf{d},\mathbf{e})$	$(\mathbf{d},\mathbf{e}) \in Q(4)$	$c(\mathbf{d},\mathbf{e})$
$((0,0,0,1),(6,0,0,0))$	$+1$	$((2,1,0,0),(2,2,0,0))$	$+6$
$((1,0,1,0),(4,1,0,0))$	-2	$((4,0,0,0),(2,0,0,1))$	-1
$((0,2,0,0),(4,1,0,0))$	-1	$((4,0,0,0),(1,1,1,0))$	$+5$
$((2,1,0,0),(3,0,1,0))$	-3	$((4,0,0,0),(0,3,0,0))$	-5

In particular, the term in $\alpha_4 W_2^7$ with the largest coefficient is $6\Sigma_2^2\Sigma_3 W_2^2 W_3^2$, and the term independent of W_2 is $-5\Sigma_2^4 W_3^3$.

Corollary 3.4. *For any positive integer N, there are integers $c(\mathbf{d},\mathbf{e})$ corresponding to pairs of partitions $(\mathbf{d},\mathbf{e}) \in Q(N)$ such that the degree $2N-1$ homogeneous polynomial in $2N$ variables,*

$$F_N(u_1,\dots,u_N;v_1,\dots,v_N) := \sum_{(\mathbf{d},\mathbf{e})\in Q(N)} c(\mathbf{d},\mathbf{e}) \prod_{r=1}^{N} u_r^{d_r} \prod_{s=1}^{N} v_s^{e_s},$$

when evaluated at $u_r = \Sigma_{r+1}(\mathbf{a}), v_s = W_{s+1}(\mathbf{a}), 1 \le r,s \le N$, takes the value

$$F_N(\Sigma_2(\mathbf{a}),\dots,\Sigma_{N+1}(\mathbf{a}); W_2(\mathbf{a}),\dots,W_{N+1}(\mathbf{a})) = \alpha_N W_2(\mathbf{a})^{2N-1}.$$

From the $2N$ variables $u_r, v_s (1 \le r,s \le N)$ let us form $2N-1$ “rational” variables:

$$\rho_r := \frac{u_r}{v_r} \quad (1 \le r \le N) \quad \text{and} \quad \tau_s := \frac{v_{s+1}}{v_1} \quad (1 \le s \le N-1).$$

Then $F_1(u_1;v_1) = u_1 = \rho_1 v_1$ and $F_2(u_1,u_2;v_1,v_2) = u_2v_1^2 - u_1^2 v_2 = (\rho_2 - \rho_1^2)v_1^2 v_2$, whence

$$\frac{F_1(u_1;v_1)}{v_1} = \rho_1 \quad \text{and} \quad \frac{F_2(u_1,u_2;v_1,v_2)}{v_1^3} = (\rho_2 - \rho_1^2)\tau_1.$$

A corresponding identity can be obtained for each F_N. Indeed if $N \ge 2$ then

$$c_{N,N} = \alpha_1^N W_{N+1} + \sum_{m=2}^{N-1} \left(\sum_{\mathbf{d}\in P(N,m)} \binom{m}{\mathbf{d}} \alpha(\mathbf{d}) \right) W_{m+1},$$

by Theorem 3.2, whence

$$F_N(u_1,\dots,u_N;v_1,\dots,v_N) = u_N v_1^{2N-2} - u_1^N v_1^{N-2} v_N + R_N,$$

where $R_N = R_N(u_1,\dots,u_{N-1};v_1,\dots,v_{N-1})$ is a polynomial which does not involve the variables u_N and v_N. Now

$$u_N v_1^{2N-2} - u_1^N v_1^{N-2} v_N = (\rho_N - \rho_1^N)\tau_{N-1} v_1^{2N-1},$$

whence induction on N utilizing Theorem 3.2 and Corollary 3.4 establishes the general identity for F_N in Corollary 3.5 below. Once again, some additional notation allows us to express the result compactly. For any $\mathbf{d} \in (\mathbb{Z}^+)^N$ and $\mathbf{e} \in (\mathbb{Z}^+)^{N-1}$ we write

$$\rho(\mathbf{d}) := \prod_{r=1}^{N} \rho_r^{d_r} \quad \text{and} \quad \tau(\mathbf{e}) := \prod_{s=1}^{N-1} \tau_s^{e_s}.$$

Also for any partition pair $(\mathbf{d},\mathbf{e}) \in Q(N)$ note that the sequence $\mathbf{d}^* \in (\mathbb{Z}^+)^{N-1}$ given by $\mathbf{d}^* := \Omega_{N-1}(\mathbf{d}+\mathbf{e})$ is a length $N-1$ partition of $N-1$, since

$$\begin{aligned}\sum_{r=1}^{N-1} r d_r^* &= \sum_{r=2}^{N}(r-1)(d_r+e_r)\\ &= \sum_{r=1}^{N} r d_r + \sum_{r=1}^{N} r e_r - \sum_{r=1}^{N}(d_r+e_r)\\ &= N + (2N-2) - (2N-1) = N-1.\end{aligned}$$

Let $P(N)$ be the set of all length N partitions of N, and for each $\mathbf{d}^* \in P(N-1)$, let us define the subfamily of partition pairs $Q^*(\mathbf{d}^*) := \{(\mathbf{d},\mathbf{e}) \in Q(N) : \Omega_{N-1}(\mathbf{d}+\mathbf{e}) = \mathbf{d}^*\}$. Then induction establishes

Corollary 3.5. *For any integer $N \geq 2$, the polynomial F_N satisfies the identity*

$$\begin{aligned}\frac{F_N(u_1,\ldots,u_N;v_1,\ldots,v_N)}{v_1^{2N-1}} &= \sum_{(\mathbf{d},\mathbf{e})\in Q(N)} c(\mathbf{d},\mathbf{e})\rho(\mathbf{d})\tau(\Omega_{N-1}(\mathbf{d}+\mathbf{e}))\\ &= \sum_{\mathbf{d}^*\in P(N-1)} f_{\mathbf{d}^*}(\rho_1,\ldots,\rho_N)\tau(\mathbf{d}^*),\end{aligned}$$

where

$$f_{\mathbf{d}^*}(\rho_1,\ldots,\rho_N) := \sum_{(\mathbf{d},\mathbf{e})\in Q^*(\mathbf{d}^*)} c(\mathbf{d},\mathbf{e})\rho(\mathbf{d}).$$

In particular we have

$$f_{(1)} = \rho_2 - \rho_1^2,$$

$$f_{(2,0)} = 2\rho_1^3 - 2\rho_1\rho_2, \qquad f_{(0,1)} = \rho_3 - \rho_1^3,$$

$$f_{(3,0,0)} = 6\rho_1^2\rho_2 - \rho_2^2 - 5\rho_1^4, \quad f_{(1,1,0)} = 5\rho_1^4 - 3\rho_1^2\rho_2 - 2\rho_1\rho_3, \quad f_{(0,0,1)} = \rho_4 - \rho_1^4.$$

For each $\mathbf{d}^* \in P(N-1)$, the polynomial $f_{\mathbf{d}^*}$ in the N variables $\rho_1,\ldots,\rho_N$ has coefficients which are a subfamily of the coefficients introduced in Theorem 3.3. Each $(\mathbf{d},\mathbf{e}) \in Q(N)$ determines a unique $\mathbf{d}^* = \Omega_{N-1}(\mathbf{d}+\mathbf{e}) \in P(N-1)$, so the families $\{c(\mathbf{d},\mathbf{e}) : (\mathbf{d},\mathbf{e}) \in Q^*(\mathbf{d}^*)\}$ comprise a partition of the family of coefficients $\{c(\mathbf{d},\mathbf{e}) : (\mathbf{d},\mathbf{e}) \in Q(N)\}$ introduced in Theorem 3.3. For each $\mathbf{d}^* \in P(N-1)$, let $\mathbf{e} \in (\mathbb{Z}^+)^N$ be such that $\Omega_{N-1}(\mathbf{e}) = \mathbf{d}^*$ and $e_1 = \Sigma_{r=1}^{N-1} d_r^*$. Then $\mathbf{e}$ is a length N partition of $2N-2$ with weight $2N-1$, and $(\mathbf{d},\mathbf{e}) \in Q^*(\mathbf{d}^*)$ when $\mathbf{d} = (N,0,\ldots,0)$. Hence $f_{\mathbf{d}^*}$ contains the term $c(\mathbf{d},\mathbf{e})\rho_1^N$, and it follows that $f_{\mathbf{d}^*}$ is of degree N. In particular our earlier calculations show that $f_{\mathbf{d}^*}(\rho_1,\ldots,\rho_N) = \rho_N - \rho_1^N$ when $\mathbf{d}^* = (0,\ldots,0,1) \in P(N-1)$, and when evaluated at $\rho_1 = \ldots = \rho_N = 1$ this polynomial takes the value 0, so the sum of its coefficients is 0. Then induction establishes

Corollary 3.6. *For any $N \geq 2$ and each $\mathbf{d}^* \in P(N-1)$, the polynomial $f_{\mathbf{d}^*}(\rho_1,\ldots,\rho_N)$ is of degree N, and the sum of its coefficients is 0.*

Since each $f_{\mathbf{d}^*}$ satisfies $f_{\mathbf{d}^*}(1,\ldots,1) = 0$, it immediately follows that we have

Corollary 3.7. *For any integer $N \geq 2$, the polynomial $F_N(u_1,\ldots,u_N;v_1,\ldots,v_N)$ has sum of coefficients equal to 0, and in fact satisfies the identity $F_N(u_1,\ldots,u_N;u_1,\ldots,u_N) = 0$.*

4. PARTICULAR EVALUATIONS OF THE ASYMPTOTIC EXPANSION

We shall now consider evaluations of the coefficient sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of the asymptotic expansion of the equipoise curve $E_1(\mathbf{a})$, for particular choices of $\mathbf{a} \in (\mathbb{R}^+)^n$.

First, note that $\Sigma_{N+1}(\mathbf{a}) = 0 = W_{N+1}(\mathbf{a})$ for all $N \geq n$. This causes no concern if we use Corollary 3.4 to determine α_N by evaluating F_N at $u_r = \Sigma_{r+1}(\mathbf{a}), v_s = W_{s+1}(\mathbf{a}), 1 \leq r, s \leq N$. On the other hand, it is not immediately obvious how we should use Corollary 3.5 to determine α_N when $N \geq n$, since the rational variable $\rho_r = u_r/v_r$ does not have a stand-alone value when $u_r = 0, v_r = 0$. However, for each partition pair $(\mathbf{d}, \mathbf{e}) \in Q(N)$ the product $\rho(\mathbf{d})\tau(\Omega_{N-1}(\mathbf{d}+\mathbf{e}))$ contains the factor $\rho_r^{d_r}\tau_{r-1}^{d_r+e_r} = u_r^{d_r}v_r^{e_r}/v_1^{d_r+e_r}$, which takes the value 0 when $u_r = 0$ and $v_r = 0$, since $\Omega_{n-1}(\mathbf{a})$ nonzero ensures that $v_1 = W_2(\mathbf{a})$ is nonzero. Thus, Corollary 3.5 yields α_N as the value of F_N/v_1^{2N-1} when $u_r = \Sigma_{r+1}(\mathbf{a}), v_s = W_{s+1}(\mathbf{a}), 1 \leq r, s \leq N$, by noting that the only products $f_{\mathbf{d}^*}\tau(\mathbf{d}^*)$ that can be nonzero correspond to partitions $\mathbf{d}^* \in P(N-1)$ with $\Omega_{N-n}(\mathbf{d}^*) = \mathbf{0}$.

Example 4.1. If $\mathbf{a} \in (\mathbb{R}^+)^2$ with $\Omega_1(\mathbf{a})$ nonzero, it is trivial to verify that the equipoise curve $E_1(\mathbf{a})$ is the straight line $y = x + \alpha_1$, with $\alpha_1 = a_1$ and $\alpha_N = 0$ for $N \geq 2$.

Example 4.2. If $\mathbf{a} \in (\mathbb{R}^+)^3$ with $\Omega_2(\mathbf{a})$ nonzero, the equipoise curve $E_1(\mathbf{a})$ is a hyperbola with asymptote $y = x + \alpha_1$. Corollary 3.5 yields α_N as the value of $f_{\mathbf{d}^*}\tau(\mathbf{d}^*)$ for $\mathbf{d}^* = (N-1, 0, \ldots, 0) \in P(N-1)$, with the evaluation $\rho_1 = \Sigma_2(\mathbf{a})/W_2(\mathbf{a}), \rho_2 = \Sigma_3(\mathbf{a})/W_3(\mathbf{a})$ and $\tau_1 = W_3(\mathbf{a})/W_2(\mathbf{a})$, noting that any products of ρ_2 and τ_1 are equal to zero if $a_3 = 0$. We have

$$\alpha_1 = \rho_1, \quad \alpha_2 = (\rho_2 - \rho_1^2)\tau_1, \quad \alpha_3 = (2\rho_1^3 - 2\rho_1\rho_2)\tau_1^2, \quad \alpha_4 = (6\rho_1^2\rho_2 - \rho_2^2 - 5\rho_1^4)\tau_1^3,$$

and so on. However we do not explicitly know the coefficients of $f_{(N-1,0,\ldots,0)}$ in general. On the other hand, Theorem 3.2 conveniently determines α_N recursively in this case. In fact, with α_1 and α_2 as determined above, we have all later terms given by the recurrence

$$\alpha_N = -\frac{W_3(\mathbf{a})}{W_2(\mathbf{a})}\left(\sum_{s=1}^{N-1}\alpha_s\alpha_{N-s}\right) \qquad \text{for } N \geq 3.$$

The substitution $\beta_N := -\tau_1\alpha_N$ for $N \geq 1$ converts the recurrence to a pure convolution

$$\beta_N = \sum_{s=1}^{N-1}\beta_s\beta_{N-s} \qquad \text{for } N \geq 3,$$

corresponding to the classical recurrence satisfied by Catalan numbers, but now with initial conditions $\beta_1 = -\rho_1\tau_1$ and $\beta_2 = (\rho_1^2 - \rho_2)\tau_1^2$. There is an extensive literature on Catalan numbers. An accessible and readily available discussion is the subject of Chapter 7 of [3].

Introducing the generating function

$$F(z) := \beta_1 z + \beta_2 z^2 + \ldots + \beta_N z^N + \ldots = \sum_{r=1}^{\infty}\beta_r z^r,$$

and letting $Z := \tau_1 z$, we find that F satisfies $F(z)^2 - F(z) - Z(\rho_1 + \rho_2 Z) = 0$, so

$$F(z) = \frac{1 - \sqrt{1 + 4Z(\rho_1 + \rho_2 Z)}}{2}.$$

Binomial series expansion now leads to an explicit closed-form solution for β_N, whence

$$\alpha_N = (-\tau_1)^{N-1}\sum_{s=0}^{\lfloor N/2\rfloor}\frac{(-1)^s}{2N-2s-1}\binom{2N-2s-1}{N-2s, s, N-s-1}\rho_1^{N-2s}\rho_2^s \qquad \text{for } N \geq 1.$$

Here the quotient of the trinomial coefficient with $2N-2s-1$ can be regarded as a generalized Catalan number. For example, this explicit solution readily yields

$$\alpha_5 = \tau_1^4 \left[\frac{1}{9} \binom{9}{5,0,4} \rho_1^5 - \frac{1}{7} \binom{7}{3,1,3} \rho_1^3 \rho_2 + \frac{1}{5} \binom{5}{1,2,2} \rho_1 \rho_2^2 \right],$$

whence $\alpha_5 = (14\rho_1^5 - 20\rho_1^3\rho_2 + 6\rho_1\rho_2^2)\tau_1^4$. Note by Corollary 3.6 that each $f_{\mathbf{d}^*}$ has coefficient sum zero, so the generalised Catalan numbers have zero alternating sum for $N \geq 2$:

$$\sum_{s=0}^{\lfloor N/2 \rfloor} \frac{(-1)^s}{2N-2s-1} \binom{2N-2s-1}{N-2s,s,N-s-1} = 0.$$

Alternatively, this identity can be deduced from the quadratic identity for $F(z)$ when $\rho_1^2 = \rho_2$. Putting $Z^* := \rho_1\tau_1 z$ then gives $F(z)^2 - F(z) - Z^*(1+Z^*) = 0$, and binomial series expansion of the solution yields the zero alternating sum noted.

Example 4.3. Let us now consider the constant sequence $\mathbf{a} = \mathbf{1} \in (\mathbb{R}^+)^n$ in which each term is equal to 1. Then $\Sigma_k(\mathbf{1}) = \binom{n}{k} = W_k(\mathbf{1})$ for $1 \leq k \leq n$, so in this case $\alpha_1 = 1$ and Corollaries 3.4 and 3.7 imply that $\alpha_N = 0$ for $N \geq 2$. Hence, as in Example 4.1, the equipoise curve $E_1(\mathbf{1})$ is the straight line $y = x+1$. This is confirmed by noting that $\Delta P_{\mathbf{1}}(x,1) = (x+1)^n - x^n$ and $S_{\mathbf{1}}(x,y) = (x^n - y^n)/(x-y)$, so $\Delta P_{\mathbf{1}}(x,1) = S_{\mathbf{1}}(x,y)$ holds when $y = x+1$.

Example 4.4. Let $\delta := (\delta_1, \delta_2, \ldots, \delta_n) \in \mathbb{R}^n$ be a sequence in which every term satisfies $|\delta_r| \leq \epsilon$ for some small strictly positive $\epsilon \in \mathbb{R}^+$. Then $\mathbf{a} := \mathbf{1} + \delta \in (\mathbb{R}^+)^n$ is a small perturbation of the constant sequence $\mathbf{1}$. Let $\Sigma_0(\delta) := 1$. Then for each $k \geq 1$ we have

$$\Sigma_k(\mathbf{1}+\delta) = \sum_{s=0}^{k} \binom{n-s}{k-s} \Sigma_s(\delta) \qquad \text{and} \qquad W_k(\mathbf{1}+\delta) = \binom{n}{k} + W_k(\delta).$$

It is convenient to scale the functions Σ_k and W_k by dividing by $\Sigma_k(\mathbf{1}) = W_k(\mathbf{1}) = \binom{n}{k}$ when $1 \leq k \leq n$: for all $\mathbf{a} \in (\mathbb{R}^+)^n$ we define

$$\Sigma_k^*(\mathbf{a}) := \Sigma_k(\mathbf{a}) / \Sigma_k(\mathbf{1}) \qquad \text{and} \qquad W_k^*(\mathbf{a}) := W_k(\mathbf{a}) / W_k(\mathbf{1}).$$

(It can easily be shown that $\Sigma_k^*(\mathbf{a})$ is the expected value of the product of terms in a k-term subsequence of $\mathbf{a}$, and $W_k^*(\mathbf{a})$ is the expected value of the last term in a k-term subsequence of $\mathbf{a}$.) It is also appropriate to define $\Sigma_0^*(\delta) = 1$. Then for $1 \leq k \leq n$ the earlier identities become

$$\Sigma_k^*(\mathbf{1}+\delta) = \sum_{s=0}^{k} \binom{k}{s} \Sigma_s^*(\delta) \qquad \text{and} \qquad W_k^*(\mathbf{1}+\delta) = 1 + W_k^*(\delta).$$

We keep n fixed and let $\epsilon \to 0^+$, so $O(\Sigma_k^*(\delta)) = O(\epsilon^k)$ and $O(W_k^*(\delta)) = O(\epsilon)$. In particular,

$$\alpha_1 = 1 + 2\Sigma_1^*(\delta) - W_2^*(\delta) + O(\epsilon^2).$$

For any integer $s \geq 0$, put

$$\lambda_s := \binom{n}{s+2} \bigg/ \binom{n}{2} = \binom{n-2}{s} \bigg/ \binom{s+2}{2}$$

and for any $\mathbf{d} \in (\mathbb{Z}^+)^{N-1}$ define $\lambda(\mathbf{d}) := \Pi_{s=1}^{N-1} \lambda_s^{d_s}$. Now evaluating the coefficient α_N at $\mathbf{1}+\delta$ using Corollaries 3.4 and 3.5 is convenient so long as we know $f_{\mathbf{d}^*}$ explicitly for each $\mathbf{d}^* \in P(N-1)$. We have $O(f_{\mathbf{d}^*}) = O(\epsilon)$ because $\rho_r = 1 + O(\epsilon)$ for $1 \leq r \leq n-1$ and $f_{\mathbf{d}^*}$ has zero coefficient sum by Corollary 3.6. As $\tau_s = \lambda_s + O(\epsilon)$ for $1 \leq s \leq n-2$, it follows for $N \geq 2$ that

$$\alpha_N = \sum_{\mathbf{d}^* \in P(N-1)} f_{\mathbf{d}^*}(\rho_1, \ldots, \rho_N) \lambda(\mathbf{d}^*) + O(\epsilon^2)$$

where

$$f_{\mathbf{d}^*}(\rho_1,\dots,\rho_N)=\sum_{(\mathbf{d},\mathbf{e})\in Q^*(\mathbf{d}^*)} c(\mathbf{d},\mathbf{e})\left(N\Sigma_1^*(\delta)+\sum_{r=1}^{N} d_r\left[\Sigma_1^*(\delta)-W_{r+1}^*(\delta)\right]\right)+O(\epsilon^2).$$

On the other hand, using Theorem 3.2 to evaluate α_N at $\mathbf{1}+\delta$ yields α_1 as above and for $N\geq 2$ yields α_N via the recurrence

$$\alpha_N=-\sum_{s=1}^{N-2}(s+1)\lambda_s\alpha_{N-s}-N\lambda_{N-1}\left(\Sigma_1^*(\delta)-W_2^*(\delta)\right)+\lambda_{N-1}\left(\Sigma_1^*(\delta)-W_{N+1}^*(\delta)\right)+O(\epsilon^2).$$

It follows by induction for $N\geq 2$ that α_N has zero sum for the coefficients of the family of functions $\Sigma_1^*(\delta)$ and $W_k^*(\delta)$ with $2\leq k\leq N+1$.

Note in particular the special case in which $A_{n-1}(\delta)=\mathbf{0}$, so $\delta_r=0$ for $1\leq r\leq n-1$ and $|\delta_n|\leq\epsilon$. Then $\Sigma_1^*(\delta)=\delta_n/n$ and $W_k^*(\delta)=k\delta_n/n$ for $2\leq k\leq n$, so $\alpha_1=1$ and $\alpha_N=0$ for $N\geq 2$. This is confirmed directly by checking that $\Delta P_{\mathbf{1}+\delta}(x,1)=(x+1)^n-x^n+\delta_n(x+1)^{n-1}$ and $S_{\mathbf{1}+\delta}(x,y)=(x^n-y^n)/(x-y)+\delta_n y^{n-1}$ are equal precisely when $y=x+1$.

5. SELECTED INEQUALITIES

To conclude, let us briefly sample some of the inequalities between the polynomials $\Delta P_{\mathbf{a}}(x,1)$ and $S_{\mathbf{a}}(x,y)$ which are consequences of the preceding asymptotic analysis.

Case 1. $\mathbf{a}=\mathbf{1}\in(\mathbb{R}^+)^n$ *with* $n\geq 2$.

In this case the equipoise curve is $y=x+1$, and simple but elegant inequalities are already implied by Theorems 2.2 and 2.3. For instance, $\Delta P_{\mathbf{1}}(1,1)=2^n-1$ and $S_{\mathbf{1}}(1,3)=(3^n-1)/2$. The point $(1,3)$ lies above the equipoise curve, so is in the S-region, and Theorem 2.2 implies

$$2^n-1<\frac{3^n-1}{2}.$$

Indeed, the lines $y=x+2$ and $y=x+\frac{1}{2}$ lie, respectively, in the S-region and the ΔP-region, so for $x>0$ we have

$$\left(x+\frac{1}{2}\right)^n-x^n<\frac{(x+1)^n-x^n}{2}<\frac{(x+2)^n-x^n}{4}.$$

These inequalities would usually be deduced from the convexity of $y=x^n$, and actually hold for all $x\in\mathbb{R}$ when n is even.

Since $\alpha_1=1$ and $\alpha_{N+1}=0$ for $N\geq 1$ when $\mathbf{a}=\mathbf{1}$, less familiar inequalities can be derived by noting that the curves $y=x+1+x^{-N}$ and $y=x+1-x^{-N}$ lie, respectively, in the S-region and the ΔP-region when $N\geq 1$. Thus for $x>0$ we have

$$\frac{\left(x+1-\frac{1}{x^N}\right)^n-x^n}{1-\frac{1}{x^N}}<(x+1)^n-x^n<\frac{\left(x+1+\frac{1}{x^N}\right)^n-x^n}{1+\frac{1}{x^N}}$$

Once again these inequalities could be deduced from convexity of $y=x^n$, but now their form is more naturally suggested by the asymptotic expansion of the equipoise curve.

Case 2. $\mathbf{a}=\mathbf{1}+\delta\in(\mathbb{R}^+)^n$ *with* $n\geq 3$, *and there is some small strictly positive* $\epsilon\in\mathbb{R}^+$ *such that* $|\delta_r|\leq\epsilon$ *for* $1\leq r\leq n$.

Let us consider the special case in which $\delta_1=-\epsilon$, $\delta_n=\epsilon$ and $\delta_r=0$ for $2\leq r\leq n-1$. Then

$$\Delta P_{\mathbf{1}+\delta}(x,1)=(x+1)^n-x^n-\epsilon^2(x+1)^{n-2}\quad\text{and}\quad S_{\mathbf{1}+\delta}(x,y)=\frac{y^n-x^n}{y-x}+\epsilon(y^{n-1}-x^{n-1}).$$

In this case $\Sigma_1^*(\delta) = 0, \Sigma_2^*(\delta) = -2\epsilon/n(n-1), \Sigma_k^*(\delta) = 0$ for $3 \le k \le n$, and $W_k^*(\delta) = k\epsilon/n$ for $2 \le k \le n$. From Example 4.4 we have

$$\alpha_1 = 1 - \frac{2}{n}\epsilon + O(\epsilon^2), \quad \alpha_2 = \frac{n-2}{3n}\epsilon + O(\epsilon^2), \quad \alpha_3 = -\frac{(n+1)(n-2)}{18n}\epsilon + O(\epsilon^2).$$

Then $\alpha_2 > 0$ and $\alpha_3 < 0$, so the curves $y = x + \alpha_1$ and $y = x + \alpha_1 + \alpha_2 x^{-1}$ lie, respectively, below and above the equipoise curve for sufficiently large x. Hence

$$\begin{aligned}\frac{(x+\alpha_1)^n - x^n}{\alpha_1} &+ \epsilon\left[(x+\alpha_1)^{n-1} - x^{n-1}\right] \\ &< (x+1)^n - x^n - \epsilon^2(x+1)^{n-2} \\ &< \frac{\left(x+\alpha_1+\frac{\alpha_2}{x}\right)^n - x^n}{\alpha_1 + \frac{\alpha_2}{x}} + \epsilon\left[\left(x+\alpha_1+\frac{\alpha_2}{x}\right)^{n-1} - x^{n-1}\right],\end{aligned}$$

where α_1 and α_2 take the exact values

$$\alpha_1 = \frac{\binom{n}{2} - \epsilon^2}{\binom{n}{2} + (n-1)\epsilon} \quad \text{and} \quad \alpha_2 = \frac{\binom{n}{3} - (n-2)\epsilon^2 - \alpha_1^2\left[\binom{n}{3} + \binom{n-1}{2}\epsilon\right]}{\binom{n}{2} + (n-1)\epsilon}.$$

Case 3. $\mathbf{a} := (a,b,c) \in (\mathbb{R}^+)^3$ *with* $\Omega_2(\mathbf{a}) = (b,c) \ne (0,0)$.

As shown in Example 4.2, for $N \ge 1$ each α_N is a function of ρ_1, ρ_2 and τ_1 in this case. If $c = 0$, we easily verify that $\alpha_1 = a$ and $\alpha_N = 0$ for $N \ge 2$. Now suppose $c > 0$. Then ρ_1, ρ_2 and τ_1 have stand-alone values, and $\alpha_2 = 0$ precisely when $\rho_2 = \rho_1^2$. But then $\alpha_N = 0$ for $N \ge 2$, by the previously noted alternating sum identity for the generalised Catalan numbers. If $\rho_2 > \rho_1^2$ then $\alpha_2 > 0$ and the summation identity for α_N in Example 4.2 implies $(-1)^N\alpha_N > 0$, so α_N alternates in sign for $N \ge 2$. Similarly if $\rho_2 < \rho_1^2$ then $\alpha_2 < 0$ and α_N alternates in sign for $N \ge 2$. As in Case 2 above we can deduce relevant inequalities. In particular, if $\rho_2 > \rho_1^2$ then for sufficiently large x we have

$$\begin{aligned}ax^2 + bx(x+\alpha_1) + c(x+\alpha_1)^2 &< (x+a)(x+b)(x+c) - x^3 \\ &< ax^2 + bx\left(x+\alpha_1+\frac{\alpha_2}{x}\right) + c\left(x+\alpha_1+\frac{\alpha_2}{x}\right)^2,\end{aligned}$$

where α_1 and α_2 have their exact values, given explicitly in Example 4.2.

REFERENCES

[1] T.J. BROMWICH, *An Introduction to the Theory of Infinite Series*, Chap. 6 (MacMillan, 1965).

[2] R.B. EGGLETON and W.P. GALVIN, A polynomial inequality generalising an integer inequality, *J. Inequal. Pure and Appl. Math.*, **3** (2002), Art. 52, 9 pp. [ONLINE: `http://jipam.vu.edu.au/v3n4/043_02.html`]

[3] P. HILTON, D. HOLTON and J. PEDERSEN, *Mathematical Vistas – From a Room with Many Windows,* Springer (2002).

[4] K. WEIERSTRASS, Über die Theorie der analytischen Facultäten, *Crelle's Journal,* **51** (1856), 1–60.