Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 1, Article 8, 2006

SHARP ERROR BOUNDS FOR SOME QUADRATURE FORMULAE AND APPLICATIONS

NENAD UJEVIĆ

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SPLIT
TESLINA 12/III, 21000 SPLIT
CROATIA.
ujevic@pmfst.hr

Received 26 May, 2004; accepted 10 November, 2005
Communicated by C.E.M. Pearce

ABSTRACT. In the article "N. Ujević, A generalization of the pre-Grüss inequality and applications to some quadrature formulae, *J. Inequal. Pure Appl. Math.*, **3**(2), Art. 13, 2002" error bounds for some quadrature formulae are established. Here we prove that all inequalities (error bounds) obtained in this article are sharp. We also establish a new sharp averaged midpoint-trapezoid inequality and give applications in numerical integration.

Key words and phrases: Sharp error bounds, Quadrature formulae, Numerical integration.

2000 *Mathematics Subject Classification.* Primary 26D10, Secondary 41A55.

1. INTRODUCTION

In recent years a number of authors have considered error inequalities for some known and some new quadrature rules. For example, this topic is considered in [1] – [6] and [11] – [14].

In this paper we consider the midpoint, trapezoid and averaged midpoint-trapezoid quadrature rules. These rules are also considered in [12], where some new improved versions of the error inequalities for the mentioned rules are derived.

Here we first prove that all inequalities obtained in [12] are sharp. Second, we specially consider the averaged midpoint-trapezoid quadrature rule. In [6] it is shown that the last mentioned rule has a better estimation of error than the well-known Simpson's rule and in [13] it is shown that this rule is an optimal quadrature rule. We give a new sharp error bound for this rule. Finally, we give applications in numerical integration.

ISSN (electronic): 1443-5756

103-04

2. MIDPOINT INEQUALITY

Let $I \subset \mathbb{R}$ be a closed interval and $a, b \in \operatorname{Int} I$, $a < b$. Let $f : I \to \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_2(a,b)$. We define the mapping

$$\Phi(t) = \begin{cases} t - \frac{2a+b}{3}, & t \in \left[a, \frac{a+b}{2}\right] \\ t - \frac{a+2b}{3}, & t \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

such that $\Phi_0(t) = \Phi(t) / \|\Phi\|_2$, where

$$\|\Phi\|_2^2 = \int_a^b (\Phi(t))^2 \, dt = \frac{(b-a)^3}{36}.$$

We have

$$\begin{aligned} Q(f;a,b) &= \int_a^b \Phi_0(t) f'(t) dt \\ &= \frac{2}{\sqrt{b-a}} \left[f(a) + f\left(\frac{a+b}{2}\right) + f(b) - \frac{3}{b-a} \int_a^b f(t) dt \right]. \end{aligned}$$

In [12] we can find the following midpoint inequality

$$\left| f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t) dt \right| \le \frac{(b-a)^{3/2}}{2\sqrt{3}} C_1, \tag{2.1}$$

where

$$C_1 = \left\{ \|f'\|_2^2 - \frac{[f(b)-f(a)]^2}{b-a} - [Q(f;a,b)]^2 \right\}^{\frac{1}{2}}. \tag{2.2}$$

Proposition 2.1. *The inequality (2.1) is sharp in the sense that the constant* $\frac{1}{2\sqrt{3}}$ *cannot be replaced by a smaller one.*

Proof. We first define the mapping

$$f(t) = \begin{cases} \frac{1}{2}t^2, & t \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2}t^2 - t + \frac{1}{2}, & t \in \left(\frac{1}{2}, 1\right] \end{cases} \tag{2.3}$$

and note that f is a Lipschitzian function.

On the other hand, each Lipschitzian function is an absolutely continuous function [10, p. 227].

Let us now assume that the inequality (2.1) holds with a constant $C > 0$, i.e.

$$\left| f\left(\frac{a+b}{2}\right)(b-a) - \int_a^b f(t) dt \right| \le C (b-a)^{3/2} C_1, \tag{2.4}$$

where C_1 is defined by (2.2). Choosing $a = 0$, $b = 1$ and f defined by (2.3), we get

$$\int_0^1 f(t) dt = \frac{1}{24}, \ f\left(\frac{1}{2}\right) = \frac{1}{8}$$

such that the left-hand side of (2.4) becomes

$$L.H.S.(2.4) = \frac{1}{12}. \tag{2.5}$$

We also find that $C_1 = \frac{1}{2\sqrt{3}}$ such that the right-hand side of (2.4) becomes

$$R.H.S.(2.4) = \frac{C}{2\sqrt{3}}. \tag{2.6}$$

From (2.4) – (2.6) we get $C \geq \frac{1}{2\sqrt{3}}$, proving that $C = \frac{1}{2\sqrt{3}}$ is the best possible in (2.1). □

3. TRAPEZOID INEQUALITY

Let $I \subset \mathbb{R}$ be a closed interval and $a, b \in \operatorname{Int} I$, $a < b$. Let $f : I \to \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_2(a, b)$. We define the mapping

$$\chi(t) = \begin{cases} t - \frac{5a+b}{6}, & t \in \left[a, \frac{a+b}{2}\right] \\ t - \frac{a+5b}{6}, & t \in \left(\frac{a+b}{2}, b\right] \end{cases}$$

such that $\chi_0(t) = \chi(t)/\left\|\chi\right\|_2$, where

$$\|\chi\|_2^2 = \int_a^b (\chi(t))^2\, dt = \frac{(b-a)^3}{36}.$$

We have

$$\begin{aligned} P(f; a, b) &= \int_a^b \chi_0(t) f'(t) dt \\ &= \frac{1}{\sqrt{b-a}} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) - \frac{6}{b-a} \int_a^b f(t) dt \right]. \end{aligned}$$

In [12] we can find the following trapezoid inequality:

$$\left| \frac{f(a) + f(b)}{2}(b-a) - \int_a^b f(t)dt \right| \leq \frac{(b-a)^{3/2}}{2\sqrt{3}} C_2, \tag{3.1}$$

where

$$C_2 = \left\{ \|f'\|_2^2 - \frac{[f(b) - f(a)]^2}{b-a} - [P(f; a, b)]^2 \right\}^{\frac{1}{2}}. \tag{3.2}$$

Proposition 3.1. *The inequality (3.1) is sharp in the sense that the constant* $\frac{1}{2\sqrt{3}}$ *cannot be replaced by a smaller one.*

Proof. We define the mapping

$$f(t) = \frac{1}{2}t^2 - \frac{1}{2}t. \tag{3.3}$$

It is obvious that f is an absolutely continuous function. Let us now assume that the inequality (3.1) holds with a constant $C > 0$, i.e.

$$\left| \frac{f(a) + f(b)}{2}(b-a) - \int_a^b f(t)dt \right| \leq C(b-a)^{3/2} C_2, \tag{3.4}$$

where C_2 is defined by (3.2).

Choosing $a = 0$, $b = 1$ and f defined by (3.3), we get

$$\int_0^1 f(t)dt = \frac{1}{12} \quad \text{and} \quad f(0) = f(1) = 0.$$

Thus, the left-hand side of (3.4) becomes

$$L.H.S.(3.4) = \frac{1}{12}. \tag{3.5}$$

The right-hand side of (3.4) becomes

$$R.H.S.(3.4) = \frac{C}{2\sqrt{3}}. \tag{3.6}$$

From (3.4) – (3.6) we get $C \geq \frac{1}{2\sqrt{3}}$, proving that $\frac{1}{2\sqrt{3}}$ is the best possible in (3.1). □

4. AVERAGED MIDPOINT-TRAPEZOID INEQUALITY

Let $I \subset \mathbb{R}$ be a closed interval and $a, b \in \operatorname{Int} I$, $a < b$. Let $f : I \to \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_2(a, b)$. We now consider a simple quadrature rule of the form

$$\begin{aligned} (4.1) \quad & \frac{f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)}{4}(b-a) - \int_a^b f(t)dt \\ &= \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right](b-a) - \int_a^b f(t)dt = R(f). \end{aligned}$$

It is not difficult to see that (4.1) is a convex combination of the midpoint quadrature rule and the trapezoid quadrature rule. In [6] it is shown that (4.1) has a better estimation of error than the well-known Simpson's quadrature rule (when we estimate the error in terms of the first derivative f' of integrand f). In [12] the following inequality is proved

$$(4.2) \qquad \left|\frac{f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)}{4}(b-a) - \int_a^b f(t)dt\right| \leq \frac{(b-a)^{3/2}}{4\sqrt{3}} C_3,$$

where

$$(4.3) \qquad C_3 = \left[\|f'\|_2^2 - \frac{[f(b)-f(a)]^2}{b-a} - \frac{1}{b-a}\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)^2\right]^{\frac{1}{2}}.$$

Proposition 4.1. *The inequality (4.2) is sharp in the sense that the constant $\frac{1}{4\sqrt{3}}$ cannot be replaced by a smaller one.*

Proof. We first define the mapping

$$(4.4) \qquad f(t) = \begin{cases} \frac{1}{2}t^2 - \frac{1}{4}t, & t \in \left[0, \frac{1}{2}\right] \\ \frac{1}{2}t^2 - \frac{3}{4}t + \frac{1}{4}, & t \in \left(\frac{1}{2}, 1\right] \end{cases}$$

and note that f is a Lipschitzian function.

Let us now assume that the inequality (4.2) holds with a constant $C > 0$, i.e.

$$(4.5) \qquad \left|\frac{f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)}{4}(b-a) - \int_a^b f(t)dt\right| \leq C(b-a)^{3/2} C_3,$$

where C_3 is defined by (4.3). Choosing $a = 0$, $b = 1$ and f defined by (4.4), we get

$$\int_0^1 f(t)dt = -\frac{1}{48}, \; f(0) = f(1) = f\left(\frac{1}{2}\right) = 0$$

such that the left-hand side of (4.5) becomes

$$(4.6) \qquad L.H.S.(4.5) = \frac{1}{48}.$$

We also find that $C_3 = \frac{1}{4\sqrt{3}}$ such that the right-hand side of (4.5) becomes

$$(4.7) \qquad R.H.S.(4.5) = \frac{C}{4\sqrt{3}}.$$

From (4.5) – (4.7) we get $C \geq \frac{1}{4\sqrt{3}}$, proving that $C = \frac{1}{4\sqrt{3}}$ is the best possible in (4.2). □

5. A SHARP ERROR INEQUALITY

In [12] we can find the following inequality

$$S(f,g)^2 \le S(f,f)S(g,g), \tag{5.1}$$

where

$$S(f,g) = \int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \int_a^b g(t)dt - \frac{1}{\|\Psi\|^2}\int_a^b f(t)\Psi(t)dt \int_a^b g(t)\Psi(t)dt \tag{5.2}$$

and Ψ satisfies

$$\int_a^b \Psi(t)dt = 0, \tag{5.3}$$

while

$$\|\Psi\|^2 = \int_a^b \Psi^2(t)dt.$$

In [14] we can find a variant of the following lemma.

Lemma 5.1. *Let $f \in C^1[a,c]$, $g \in C^1[c,b]$ be such that $f(c) = g(c)$. Then*

$$h(t) = \begin{cases} f(t), & t \in [a,c] \\ g(t), & t \in [c,b] \end{cases}$$

is an absolutely continuous function.

Theorem 5.2. *Let $f : [0,1] \to \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_2(0,1)$. Then*

$$\left|\int_0^1 f(t)dt - \frac{1}{4}\left[f(0) + 2f\left(\frac{1}{2}\right) + f(1)\right]\right| \le \frac{1}{4\sqrt{3}}\sqrt{\|f'\|^2 - 2\left[f\left(\frac{1}{2}\right) - f(0)\right]^2 - 2\left[f(1) - f\left(\frac{1}{2}\right)\right]^2}. \tag{5.4}$$

The inequality (5.4) is sharp in the sense that the constant $\frac{1}{4\sqrt{3}}$ cannot be replaced by a smaller one.

Proof. We define the functions

$$p(t) = \begin{cases} t - \frac{1}{4}, & t \in \left[0, \frac{1}{2}\right) \\ t - \frac{3}{4}, & t \in \left[\frac{1}{2}, 1\right] \end{cases} \tag{5.5}$$

and

$$\Psi(t) = \begin{cases} t, & t \in \left[0, \frac{1}{2}\right) \\ t - 1, & t \in \left[\frac{1}{2}, 1\right] \end{cases}. \tag{5.6}$$

It is not difficult to verify that

$$\int_0^1 p(t)dt = \int_0^1 \Psi(t)dt = 0. \tag{5.7}$$

We also have

$$\|p\|^2 = \int_0^1 p^2(t)dt = \frac{1}{48}, \tag{5.8}$$

$$\|\Psi\|^2 = \int_0^1 \Psi^2(t)dt = \frac{1}{12}, \tag{5.9}$$

$$\int_0^1 p(t)\Psi(t)dt = \frac{1}{48}. \tag{5.10}$$

From (5.1), (5.2) and (5.3) we get

$$\begin{aligned} &\left[\int_0^1 p(t)f'(t)dt - \frac{1}{\|\Psi\|^2}\int_0^1 p(t)\Psi(t)dt\int_0^1 f'(t)\Psi(t)dt\right]^2 \\ &\qquad \le \left[\|p\|^2 - \frac{1}{\|\Psi\|^2}\left(\int_0^1 p(t)\Psi(t)dt\right)^2\right] \\ &\qquad\quad \times \left[\|f'\|^2 - \left(\int_0^1 f'(t)dt\right)^2 - \frac{1}{\|\Psi\|^2}\left(\int_0^1 f'(t)\Psi(t)dt\right)^2\right]. \end{aligned} \tag{5.11}$$

Integrating by parts, we obtain

$$\begin{aligned} \int_0^1 p(t)f'(t)dt &= \int_0^{\frac{1}{2}}\left(t-\frac{1}{4}\right)f'(t)dt + \int_{\frac{1}{2}}^1\left(t-\frac{3}{4}\right)f'(t)dt \\ &= \frac{1}{4}\left[f(0)+2f\left(\frac{1}{2}\right)+f(1)\right] - \int_0^1 f(t)dt \end{aligned} \tag{5.12}$$

and

$$\begin{aligned} \int_0^1 f'(t)\Psi(t)dt &= \int_0^{\frac{1}{2}} tf'(t)dt + \int_{\frac{1}{2}}^1 (t-1)f'(t)dt \\ &= f\left(\frac{1}{2}\right) - \int_0^1 f(t)dt. \end{aligned} \tag{5.13}$$

We introduce the notations

$$i = \int_0^1 f(t)dt, \tag{5.14}$$

$$q = \frac{1}{4}\left[f(0)+2f\left(\frac{1}{2}\right)+f(1)\right]. \tag{5.15}$$

From (5.11) – (5.15) and (5.8) – (5.10) it follows that

$$\begin{aligned} &\left[(q-i) - \frac{1}{4}\left(f\left(\frac{1}{2}\right)-i\right)\right]^2 \\ &\qquad \le \frac{1}{64}\left[\|f'\|^2 - [f(1)-f(0)]^2 - 12\left(f\left(\frac{1}{2}\right)-i\right)^2\right] \end{aligned} \tag{5.16}$$

or

$$i^2 - 2qi + \frac{4}{3}q^2 + \frac{1}{48}\left[f(1) - f(0)\right]^2 - \|f'\|^2 + 16\left(f\left(\frac{1}{2}\right)\right)^2 - 32f\left(\frac{1}{2}\right)q \le 0. \tag{5.17}$$

If we now introduce the notations

$$\beta = -2q, \tag{5.18}$$

$$\gamma = \frac{4}{3}q^2 + \frac{1}{48}\left[f(1) - f(0)\right]^2 - \|f'\|^2 + 16\left(f\left(\frac{1}{2}\right)\right)^2 - 32f\left(\frac{1}{2}\right)q \tag{5.19}$$

then we have

$$i^2 + \beta i + \gamma \le 0. \tag{5.20}$$

Thus, $i \in [i_1, i_2]$, where

$$i_1 = \frac{-\beta - \sqrt{\beta^2 - 4\gamma}}{2}, \quad i_2 = \frac{-\beta + \sqrt{\beta^2 - 4\gamma}}{2}.$$

In other words,

$$-\frac{\beta}{2} - \frac{\sqrt{\beta^2 - 4\gamma}}{2} \le i \le -\frac{\beta}{2} + \frac{\sqrt{\beta^2 - 4\gamma}}{2}$$

or

$$\left|i + \frac{\beta}{2}\right| \le \frac{\sqrt{\beta^2 - 4\gamma}}{2}. \tag{5.21}$$

We have

$$\beta^2 - 4\gamma = \frac{1}{12}\left[\|f'\|^2 - 2\left[f\left(\frac{1}{2}\right) - f(0)\right]^2 - 2\left[f(1) - f\left(\frac{1}{2}\right)\right]^2\right]. \tag{5.22}$$

From (5.21) and (5.22) we easily find that (5.4) holds.

We have to prove that (5.4) is sharp. For that purpose, we define the function

$$f(t) = \begin{cases} \frac{1}{2}t^2 - \frac{1}{4}t + \frac{1}{32}, & t \in \left[0, \frac{1}{2}\right) \\ \frac{1}{2}t^2 - \frac{3}{4}t + \frac{9}{32}, & t \in \left[\frac{1}{2}, 1\right] \end{cases}. \tag{5.23}$$

From Lemma 5.1 we see that the above function is absolutely continuous. If we substitute the above function in the left-hand side of (5.4) then we get

$$L.H.S.(5.4) = \frac{1}{48}. \tag{5.24}$$

If we substitute the above function in the right-hand side of (5.4) then we get

$$R.H.S.(5.4) = \frac{1}{48}. \tag{5.25}$$

From (5.24) and (5.25) we conclude that (5.4) is sharp. □

Theorem 5.3. *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_2(a,b)$. Then*

$$\left| \int_a^b f(t)dt - \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{(b-a)^{3/2}}{4\sqrt{3}} \left(\|f'\|^2 - \frac{2}{b-a}\left[f\left(\frac{a+b}{2}\right) - f(a)\right]^2 - \frac{2}{b-a}\left[f(b) - f\left(\frac{a+b}{2}\right)\right]^2\right)^{\frac{1}{2}}. \tag{5.26}$$

The above inequality is sharp in the sense that the constant $1/(4\sqrt{3})$ cannot be replaced by a smaller one.

Remark 5.4. We have better estimates than (5.26). For example, we have the inequality

$$\left| \frac{b-a}{4}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] - \int_a^b f(t)dt \right| \leq \frac{1}{8}\|f'\|_\infty (b-a)^2. \tag{5.27}$$

However, note that the estimate (5.27) can be applied only if f' is bounded. On the other hand, the estimate (5.26) can be applied for absolutely continuous functions if $f' \in L_2(a,b)$.

There are many examples where we cannot apply the estimate (5.27) but we can apply (5.26).

Example 5.1. Let us consider the integral $\int_0^1 \sqrt[3]{\sin t^2}dt$. We have

$$f(t) = \sqrt[3]{\sin t^2} \quad \text{and} \quad f'(t) = \frac{2t\cos t^2}{3\sqrt[3]{\sin^2 t^2}}$$

such that $f'(t) \to \infty,\ t \to 0$ and we cannot apply the estimate (5.27). On the other hand, we have

$$\int_0^1 [f'(t)]^2 dt \leq \frac{4}{9}\max_{t\in[0,1]} \frac{t^2 \cos t^2}{\sin t^2} \int_0^1 \frac{dt}{\sqrt[3]{\sin t^2}} \leq \frac{16}{9},$$

i.e. $\|f'\|_2 \leq \frac{4}{3}$ and we can apply the estimate (5.26).

6. Applications in Numerical Integration

Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a,b]$ such that $h_i = x_{i+1} - x_i = h = (b-a)/n$. We define

$$\sigma_n(f) = \sum_{i=0}^{n-1} \left[\frac{b-a}{n}\|f'\|_2^2 - (f(x_{i+1}) - f(x_i))^2 - \left(f(x_i) - 2f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1})\right)^2\right]^{\frac{1}{2}}, \tag{6.1}$$

$$\eta_n(f) = \sum_{i=0}^{n-1} \left[\frac{b-a}{n}\|f'\|_2^2 - 2\left(f\left(\frac{x_i + x_{i+1}}{2}\right) - f(x_i)\right)^2 - 2\left(f(x_{i+1}) - f\left(\frac{x_i + x_{i+1}}{2}\right)\right)^2\right]^{\frac{1}{2}} \tag{6.2}$$

and

$$\omega_n(f) = \left[(b-a)\left\|f'\right\|_2^2 - \frac{1}{n}(f(b)-f(a))^2\right]^{\frac{1}{2}}. \tag{6.3}$$

Theorem 6.1. *Let π be a given subdivision of the interval $[a,b]$ and let the assumptions of Theorem 5.2 hold. Then*

$$\left|\int_a^b f(t)dt - \frac{h}{4}\sum_{i=0}^{n-1}\left[f(x_i) + 2f\left(\frac{x_i+x_{i+1}}{2}\right) + f(x_{i+1})\right]\right| \le \frac{b-a}{4\sqrt{3n}}\sigma_n(f) \le \frac{b-a}{4\sqrt{3n}}\omega_n(f), \tag{6.4}$$

where $\sigma_n(f)$ and $\omega_n(f)$ are defined by (6.1) and (6.3), respectively.

Proof. We have

$$\frac{h}{4}\left[f(x_i) + 2f\left(\frac{x_i+x_{i+1}}{2}\right) + f(x_{i+1})\right] - \int_{x_i}^{x_{i+1}} f(t)dt = \int_{x_i}^{x_{i+1}} K_i(t)f'(t)dt, \tag{6.5}$$

where

$$K_i(t) = \begin{cases} t - \frac{3x_i+x_{i+1}}{4}, & t \in \left[x_i, \frac{x_i+x_{i+1}}{2}\right] \\ t - \frac{x_i+3x_{i+1}}{4}, & t \in \left(\frac{x_i+x_{i+1}}{2}, x_{i+1}\right] \end{cases}.$$

From Proposition 4.1 we obtain

$$\left|\frac{h}{4}\left[f(x_i) + 2f\left(\frac{x_i+x_{i+1}}{2}\right) + f(x_{i+1})\right] - \int_{x_i}^{x_{i+1}} f(t)dt\right| \le \frac{h^{3/2}}{4\sqrt{3}}\left[\left\|f'\right\|_2^2 - \frac{1}{h}\left(f(x_{i+1}) - f(x_i)\right)^2 - \frac{1}{h}\left(f(x_i) - 2f\left(\frac{x_i+x_{i+1}}{2}\right) + f(x_{i+1})\right)^2\right]^{\frac{1}{2}}.$$

If we sum (6.5) over i from 0 to $n-1$ and apply the above inequality then we get

$$\left|\int_a^b f(t)dt - \frac{h}{4}\sum_{i=0}^{n-1}\left[f(x_i) + 2f\left(\frac{x_i+x_{i+1}}{2}\right) + f(x_{i+1})\right]\right| \le \frac{h^{3/2}}{4\sqrt{3}}\left[\sum_{i=0}^{n-1}\left\|f'\right\|_2^2 - \frac{1}{h}\left(f(x_{i+1}) - f(x_i)\right)^2 - \frac{1}{h}\left(f(x_i) - 2f\left(\frac{x_i+x_{i+1}}{2}\right) + f(x_{i+1})\right)^2\right]^{\frac{1}{2}}.$$

From the above relation and the fact $h = (b-a)/n$ we see that the first inequality in (6.4) holds.

Using the Cauchy inequality we have

$$\sum_{i=0}^{n-1}\left[\|f'\|_2^2-\frac{1}{h}\left(f(x_{i+1})-f(x_i)\right)^2\right]^{\frac{1}{2}} \le n\left[\|f'\|_2^2-\frac{1}{b-a}\sum_{i=0}^{n-1}\left(f(x_{i+1})-f(x_i)\right)^2\right]^{\frac{1}{2}} \le n\left[\|f'\|_2^2-\frac{1}{b-a}\frac{1}{n}\left(f(b)-f(a)\right)^2\right]^{\frac{1}{2}}. \tag{6.6}$$

Since

$$\|f'\|_2^2-\frac{1}{h}\left(f(x_{i+1})-f(x_i)\right)^2-\frac{1}{h}\left(f(x_i)-2f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right)^2 \le \|f'\|_2^2-\frac{1}{h}\left(f(x_{i+1})-f(x_i)\right)^2,$$

we easily conclude that the second inequality in (6.4) holds, too. □

Remark 6.2. The second inequality in (6.4) is coarser than the first inequality. It may be used to predict the number of steps needed in the compound rule for a given accuracy of the approximation. Of course, we shall use the first inequality in (6.4) to obtain the error bound. Note also that in this last case we use the same values $f(x_i)$ to calculate the approximation of the integral $\int_a^b f(t)dt$ and to obtain the error bound and recall that function evaluations are generally considered the computationally most expensive part of quadrature algorithms.

Theorem 6.3. *Under the assumptions of Theorem 6.1 we have*

$$\left|\int_a^b f(t)dt-\frac{h}{4}\sum_{i=0}^{n-1}\left[f(x_i)+2f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right]\right| \le \frac{b-a}{4\sqrt{3}n}\eta_n(f)\le\frac{b-a}{4\sqrt{3}n}\omega_n(f),$$

where $\eta_n(f)$ is defined by (6.2).

Proof. The proof of this theorem is similar to the proof of Theorem 6.1. Here we use Theorem 5.3. □

REFERENCES

[1] G.A. ANASTASSIOU, Ostrowski type inequalities, *Proc. Amer. Math. Soc.*, **123**(12) (1995), 3775–3781.

[2] P. CERONE, Three points rules in numerical integration, *Nonlinear Anal.-Theory Methods Appl.*, **47**(4) (2001), 2341–2352.

[3] P. CERONE AND S.S. DRAGOMIR, Midpoint-type rules from an inequalities point of view, *Handbook of Analytic-Computational Methods in Applied Mathematics*, Ed.: G. Anastassiou, CRC Press, New York, (2000), 135–200.

[4] P. CERONE AND S.S. DRAGOMIR, Trapezoidal-type rules from an inequalities point of view, *Handbook of Analytic-Computational Methods in Applied Mathematics*, Ed.: G. Anastassiou, CRC Press, New York, (2000), 65–134.

[5] D. CRUZ-URIBE AND C.J. NEUGEBAUER, Sharp error bounds for the trapezoidal rule and Simpson's rule, *J. Inequal. Pure Appl. Math.*, **3**(4) (2002), Art. 49. [ONLINE: http://jipam.vu.edu.au/article.php?sid=201].

[6] S.S. DRAGOMIR, P. CERONE AND J. ROUMELIOTIS, A new generalization of Ostrowski's integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means, *Appl. Math. Lett.*, **13** (2000), 19–25.

[7] A. GHIZZETTI AND A. OSSICINI, *Quadrature Formulae*, Birkhäuser Verlag, Basel/Stuttgart, 1970.

[8] A.R. KROMMER AND C.W. UEBERHUBER, *Computational Integration*, SIAM, Philadelphia, 1998.

[9] D.S. MITRINOVIĆ, J. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Acad. Publ., Dordrecht/Boston/London, 1993.

[10] I.P. NATANSON, *Theory of Functions of Real Variables*, Nauka, Moscow, 1974.

[11] C.E.M. PEARCE, J. PEČARIĆ, N. UJEVIĆ AND S. VAROŠANEC, Generalizations of some inequalities of Ostrowski-Grüss type, *Math. Inequal. Appl.*, **3**(1) (2000), 25–34.

[12] N. UJEVIĆ, A generalization of the pre-Grüss inequality and applications to some quadrature formulae, *J. Inequal. Pure Appl. Math.*, **3**(1) (2002), Art. 13. [ONLINE: http://jipam.vu.edu.au/article.php?sid=165].

[13] N. UJEVIĆ, An optimal quadrature formula of open type, *Yokohama Math. J.*, **50** (2003), 59–70.

[14] N. UJEVIĆ, Two sharp Ostrowski-like inequalities and applications, *Meth. Appl. Analysis*, **10**(3) (2004), 477–486.

[15] N. UJEVIĆ AND A.J. ROBERTS, A corrected quadrature formula and applications, *ANZIAM J.*, **45**(E) (2004), E41–E56.