

Journal of Inequalities in Pure and Applied Mathematics

NEW WEIGHTED MULTIVARIATE GRÜSS TYPE INEQUALITIES

B.G. PACHPATTE

57 Shri Niketan Colony,
Near Abhinay Talkies,
Aurangabad 431 001
(Maharashtra) India.
EMail: bgpachpatte@hotmail.com



volume 4, issue 5, article 108,
2003.

*Received 09 October, 2002;
accepted 15 December, 2003.*

Communicated by: P. Cerone

Abstract

Contents



Home Page

Go Back

Close

Quit

Abstract

In this paper we establish some new weighted multidimensional Grüss type integral and discrete inequalities by using a fairly elementary analysis .

2000 Mathematics Subject Classification: 26D15 , 26D20.

Key words: Multivariate Grüss type inequalities, Discrete inequalities, New estimates, Differentiable function, Partial derivatives, Forward difference operators, Mean value theorem .

Contents

1	Introduction	3
2	Statement of Results	4
3	Proof of Theorem 2.1	9
4	Proof of Theorem 2.2	14
	References	



New Weighted Multivariate Grüss Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 2 of 18

1. Introduction

In 1935, G. Grüss [3] proved the following classical integral inequality (see, also [4, p. 296]):

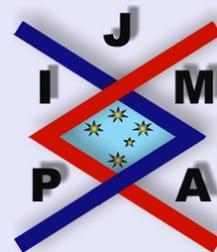
$$\left| \frac{1}{b-a} \int_a^b f(x) g(x) - \left(\frac{1}{b-a} \int_a^b f(x) dx \right) \left(\frac{1}{b-a} \int_a^b g(x) dx \right) \right| \leq \frac{1}{4} (P-p)(Q-q),$$

provided that f and g are two integrable functions on $[a, b]$ such that

$$p \leq f(x) \leq P, \quad q \leq g(x) \leq Q,$$

for all $x \in [a, b]$, where p, P, q, Q are constants.

A large number of generalizations, extensions and variants of this inequality have been given by several authors since its discovery, see [1, 2], [4] – [6] and the references given therein. The main purpose of this paper is to establish new weighted integral and discrete inequalities of the Grüss type involving functions of several independent variables. The analysis used in the proofs is elementary and our results provide new estimates on inequalities of this type.



**New Weighted Multivariate
Grüss Type Inequalities**

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 3 of 18

2. Statement of Results

In what follows, \mathbb{R} and \mathbb{N} denote the set of real and natural numbers respectively.

Let $D_i[a, b] = \{x_i : a_i < x_i < b_i\}$ for $i = 1, \dots, n$, $a_i, b_i \in \mathbb{R}$, $D = \prod_{i=1}^n D_i[a_i, b_i]$ and \bar{D} be the closure of D . For a differentiable function $u(x) : \bar{D} \rightarrow \mathbb{R}$, we denote the first order partial derivatives by $\frac{\partial u(x)}{\partial x_i}$ ($i = 1, \dots, n$) and $\int_D u(x) dx$ the n -fold integral

$$\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x_1, \dots, x_n) dx_1 \dots dx_n.$$

If

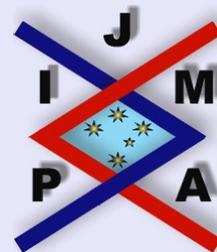
$$\left\| \frac{\partial u}{\partial x_i} \right\|_{\infty} = \sup_{x \in D} \left| \frac{\partial u(x)}{\partial x_i} \right| < \infty,$$

then we say that the partial derivatives $\frac{\partial u(x)}{\partial x_i}$ are bounded. Let $N_i[0, a_i] = \{0, 1, 2, \dots, a_i\}$, $a_i \in \mathbb{N}$, ($i = 1, \dots, n$) and $B = \prod_{i=1}^n N_i[0, a_i]$. For a function $z(x) : B \rightarrow \mathbb{R}$ we define the first order forward difference operators as

$$\begin{aligned} \Delta_1 z(x) &= z(x_1 + 1, x_2, \dots, x_n) - z(x), \dots, \Delta_n z(x) \\ &= z(x_1, \dots, x_{n-1}, x_n + 1) - z(x) \end{aligned}$$

and denote the n -fold sum over B with respect to the variable $y = (y_1, \dots, y_n) \in B$ by

$$\sum_y z(y) = \sum_{y_1=0}^{a_1-1} \cdots \sum_{y_n=0}^{a_n-1} z(y_1, \dots, y_n).$$



New Weighted Multivariate
Grüss Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 4 of 18

Clearly,

$$\sum_y z(y) = \sum_x z(x) \quad \text{for } x, y \in B.$$

If $\|\Delta_i z\|_\infty = \sup_{x \in B} |\Delta_i z(x)| < \infty$, then we say that the partial differences $\Delta_i z(x)$ are bounded. The notation

$$\sum_{t_i=y_i}^{x_i-1} \Delta_i z(y_1, \dots, y_{i-1}, t_i, x_{i+1}, \dots, x_n), \quad x_i, y_i \in N_i [0, a_i] \quad (i = 1, \dots, n),$$

we mean for $i = 1$ it is $\sum_{t_1=y_1}^{x_1-1} \Delta_1 z(t_1, x_2, \dots, x_n)$ and so on, and for $i = n$ it is $\sum_{t_n=y_n}^{x_n-1} \Delta_n \times z(y_1, \dots, y_{n-1}, t_n)$. We use the usual convention that the empty sum is taken to be zero. We use the following notations to simplify the details of presentation.

For continuous functions p, q defined on \bar{D} and differentiable on D , $w(x)$ a real-valued nonnegative and integrable function for every $x \in D$ with $\int_D w(x) dx > 0$ and $x_i, y_i \in D_i [a_i, b_i]$, we set

$$A[w, p, q] = \int_D w(x) p(x) q(x) dx - \frac{1}{\int_D w(x) dx} \left(\int_D w(x) p(x) dx \right) \left(\int_D w(x) q(x) dx \right),$$

$$H[p, x_i, y_i] = \sum_{i=1}^n \left\| \frac{\partial p}{\partial x_i} \right\|_\infty |x_i - y_i|.$$



New Weighted Multivariate Grüss Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 5 of 18

For the functions $p, q : B \rightarrow \mathbb{R}$ whose forward differences $\Delta_i p, \Delta_i q$ exist, $w(x)$ a real-valued nonnegative function defined on B and $\sum_x w(x) > 0$ and $x_i, y_i \in N_i[0, a_i]$, we set

$$L[w, p, q] = \sum_x w(x) p(x) q(x) - \frac{1}{\sum_x w(x)} \left(\sum_x w(x) p(x) \right) \left(\sum_x w(x) q(x) \right),$$

$$M[p, x_i, y_i] = \sum_{i=1}^n \|\Delta_i p\|_\infty |x_i - y_i|.$$

Our main results on weighted Grüss type integral inequalities involving functions of many independent variables are embodied in the following theorem.

Theorem 2.1. *Let f, g be real-valued continuous functions on \bar{D} and differentiable on D whose derivatives $\frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i}$ are bounded. Let $w(x)$ be a real-valued, nonnegative and integrable function for $x \in D$ and $\int_D w(x) dx > 0$. Then*

$$(2.1) \quad |A[w, f, g]| \leq \frac{1}{2 \int_D w(x) dx} \int_D w(x) \left[|g(x)| \int_D H[f, x_i, y_i] w(y) dy + |f(x)| \int_D H[g, x_i, y_i] w(y) dy \right] dx,$$



**New Weighted Multivariate
Grüss Type Inequalities**

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 6 of 18

$$(2.2) \quad |A[w, f, g]|$$

$$\leq \frac{1}{\left(\int_D w(x) dx\right)^2} \int_D w(x) \left(\int_D H[f, x_i, y_i] w(y) dy\right) \times \left(\int_D H[g, x_i, y_i] w(y) dy\right) dx.$$

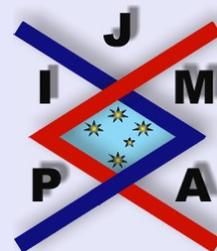
Remark 2.1. If we take $n = 1$ and $D = I = \{a < x < b\}$ in (2.1), then we get

$$\left| \int_a^b w(t) f(t) g(t) dt - \frac{1}{\int_a^b w(t) dt} \left(\int_a^b w(t) f(t) dt\right) \left(\int_a^b w(t) g(t) dt\right) \right| \leq \frac{1}{2 \int_a^b w(t) dt} \int_a^b w(t) \left[|g(t)| \int_a^b \|f'\|_\infty |t - s| w(s) ds + |f(t)| \int_a^b \|g'\|_\infty |t - s| w(s) ds \right] dt.$$

Similarly, one can obtain the special version of (2.2). It is easy to see that the upper bound given on the right side in the above inequality (when $w(t) = 1$) is different from those given by Grüss in [3].

The next theorem deals with the discrete versions of the inequalities in Theorem 2.1.

Theorem 2.2. Let f, g be real-valued functions defined on B and $\Delta_i f, \Delta_i g$ are bounded. Let $w(x)$ be a real-valued nonnegative function defined on B and



New Weighted Multivariate Grüss Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

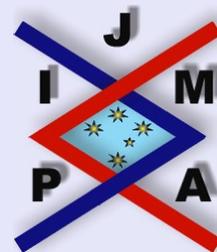
Page 7 of 18

$\sum_x w(x) > 0$. Then

$$(2.3) \quad |L[w, f, g]| \leq \frac{1}{2 \sum_x w(x)} \sum_x w(x) \left[|g(x)| \sum_y M[f, x_i, y_i] w(y) + |f(x)| \sum_y M[g, x_i, y_i] w(y) \right],$$

$$(2.4) \quad |L(w, f, g)| \leq \frac{1}{\left(\sum_x w(x)\right)^2} \sum_x w(x) \times \left(\sum_y M[f, x_i, y_i] w(y)\right) \left(\sum_y M[g, x_i, y_i] w(y)\right).$$

Remark 2.2. In a recent paper [6] the author gave multidimensional Grüss type finite difference inequalities whose proofs were based on a certain finite difference identity. Here we note that the inequalities established in (2.3) and (2.4) are of more general type and can be considered as the weighted generalizations of the similar inequalities given in [6].



New Weighted Multivariate Grüss Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 8 of 18

3. Proof of Theorem 2.1

Let $x = (x_1, \dots, x_n) \in \bar{D}$, $y = (y_1, \dots, y_n) \in D$. From the n -dimensional version of the mean value theorem we have (see [7, p. 174])

$$(3.1) \quad f(x) - f(y) = \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i)$$

and

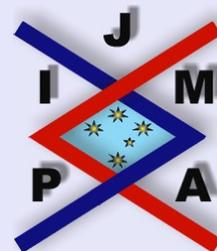
$$(3.2) \quad g(x) - g(y) = \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i),$$

where $c = (y_1 + \alpha(x_1 - y_1), \dots, y_n + \alpha(x_n - y_n))$ ($0 < \alpha < 1$). Multiplying both sides of (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively and adding we get

$$(3.3) \quad 2f(x)g(x) - g(x)f(y) - f(x)g(y) \\ = g(x) \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i) + f(x) \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i).$$

Multiplying both sides of (3.3) by $w(y)$ and integrating the resulting identity with respect to y over D we have

$$(3.4) \quad 2 \left(\int_D w(y) dy \right) f(x)g(x) \\ - g(x) \int_D w(y) f(y) dy - f(x) \int_D w(y) g(y) dy$$



New Weighted Multivariate
Grüss Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 9 of 18

$$= g(x) \int_D \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i) w(y) dy + f(x) \int_D \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i) w(y) dy.$$

Next, multiplying both sides of (3.4) by $w(x)$ and integrating the resulting identity with respect to x on D we get

$$\begin{aligned} (3.5) \quad & 2 \left(\int_D w(y) dy \right) \int_D w(x) f(x) g(x) dx \\ & - \left(\int_D w(x) g(x) dx \right) \left(\int_D w(y) f(y) dy \right) \\ & - \left(\int_D w(x) f(x) dx \right) \left(\int_D w(y) g(y) dy \right) \\ & = \int_D w(x) g(x) \left(\int_D \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i) w(y) dy \right) dx \\ & + \int_D w(x) f(x) \left(\int_D \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i) w(y) dy \right) dx. \end{aligned}$$

From (3.5) and using the properties of modulus we have

$$\begin{aligned} & |A[w, f, g]| \\ & \leq \frac{1}{2 \int_D w(x) dx} \left[\int_D w(x) |g(x)| \left(\int_D \sum_{i=1}^n \left| \frac{\partial f(c)}{\partial x_i} \right| |x_i - y_i| w(y) dy \right) dx \right. \\ & \quad \left. + \int_D w(x) |f(x)| \left(\int_D \sum_{i=1}^n \left| \frac{\partial g(c)}{\partial x_i} \right| |x_i - y_i| w(y) dy \right) dx \right] \end{aligned}$$



**New Weighted Multivariate
Grüss Type Inequalities**

B.G. Pachpatte

Title Page

Contents

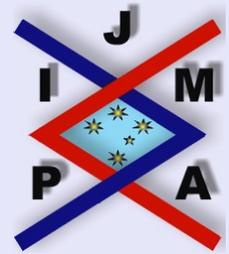


Go Back

Close

Quit

Page 10 of 18



**New Weighted Multivariate
Grüss Type Inequalities**

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 11 of 18

$$\begin{aligned} &\leq \frac{1}{2 \int_D w(x) dx} \int_D w(x) \left[|g(x)| \int_D \sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_{\infty} |x_i - y_i| w(y) dy \right. \\ &\quad \left. + |f(x)| \int_D \sum_{i=1}^n \left\| \frac{\partial g}{\partial x_i} \right\|_{\infty} |x_i - y_i| w(y) dy \right] dx \\ &= \frac{1}{2 \int_D w(x) dx} \int_D w(x) \left[|g(x)| \int_D H[f, x_i, y_i] w(y) dy \right. \\ &\quad \left. + |f(x)| \int_D H[g, x_i, y_i] w(y) dy \right] dx. \end{aligned}$$

This is the required inequality in (2.1).

Multiplying both sides of (3.1) and (3.2) by $w(y)$ and integrating the resulting identities with respect to y on D we get

$$\begin{aligned} (3.6) \quad \left(\int_D w(y) dy \right) f(x) - \int_D w(y) f(y) dy \\ = \int_D \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i) w(y) dy \end{aligned}$$

and

$$\begin{aligned} (3.7) \quad \left(\int_D w(y) dy \right) g(x) - \int_D w(y) g(y) dy \\ = \int_D \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i) w(y) dy. \end{aligned}$$

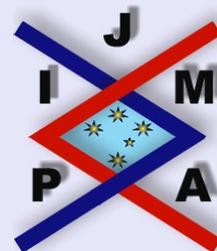
Multiplying the left sides and right sides of (3.6) and (3.7) we get

$$\begin{aligned}
 (3.8) \quad & \left(\int_D w(y) dy \right)^2 f(x) g(x) - \left(\int_D w(y) dy \right) f(x) \int_D w(y) g(y) dy \\
 & - \left(\int_D w(y) dy \right) g(x) \int_D w(y) f(y) dy \\
 & + \left(\int_D w(y) f(y) dy \right) \left(\int_D w(y) g(y) dy \right) \\
 = & \left(\int_D \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i) w(y) dy \right) \left(\int_D \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i) w(y) dy \right).
 \end{aligned}$$

Multiplying both sides of (3.8) by $w(x)$ and integrating the resulting identity with respect to x on D , by simple calculations we obtain

$$\begin{aligned}
 (3.9) \quad & \int_D w(x) f(x) g(x) dx \\
 & - \frac{1}{\int_D w(y) dy} \left(\int_D w(x) f(x) dx \right) \left(\int_D w(x) g(x) dx \right) \\
 = & \frac{1}{\left(\int_D w(y) dy \right)^2} \int_D w(x) \left(\int_D \sum_{i=1}^n \frac{\partial f(c)}{\partial x_i} (x_i - y_i) w(y) dy \right) \\
 & \times \left(\int_D \sum_{i=1}^n \frac{\partial g(c)}{\partial x_i} (x_i - y_i) w(y) dy \right) dx.
 \end{aligned}$$

From (3.9) and following the proof of the inequality (2.1) with suitable modifications we get the required inequality in (2.2). The proof is complete. \square



**New Weighted Multivariate
Grüss Type Inequalities**

B.G. Pachpatte

Title Page

Contents



Go Back

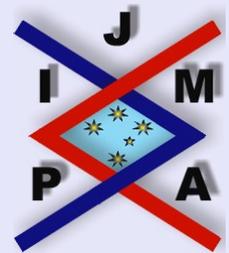
Close

Quit

Page 12 of 18

Remark 3.1. *Multiplying the left sides and right sides of (3.1) and (3.2), then multiplying the resulting identity by $w(y)$, integrating it with respect to y on D , again multiplying the resulting identity by $w(x)$, integrating it with respect to x over D and following the similar arguments as in the proofs of (2.1), (2.2) we have*

$$(3.10) \quad |A[w, f, g]| \leq \frac{1}{2 \int_D w(x) dx} \int_D w(x) \left(\int_D H[f, x_i, y_i] H[g, x_i, y_i] w(y) dy \right) dx.$$



**New Weighted Multivariate
Grüss Type Inequalities**

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 13 of 18

4. Proof of Theorem 2.2

For $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n)$ in B , it is easy to observe that the following identities hold (see [6]):

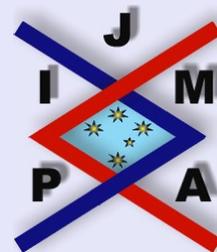
$$(4.1) \quad f(x) - f(y) = \sum_{i=1}^n \left\{ \sum_{t_i=y_i}^{x_i-1} \Delta_i f(y_1, \dots, y_{i-1}, t_i, x_{i+1}, \dots, x_n) \right\}$$

and

$$(4.2) \quad g(x) - g(y) = \sum_{i=1}^n \left\{ \sum_{t_i=y_i}^{x_i-1} \Delta_i g(y_1, \dots, y_{i-1}, t_i, x_{i+1}, \dots, x_n) \right\}.$$

Multiplying both sides of (4.1) and (4.2) by $g(x)$ and $f(x)$ respectively, and adding we obtain

$$(4.3) \quad 2f(x)g(x) - g(x)f(y) - f(x)g(y) \\ = g(x) \sum_{i=1}^n \left\{ \sum_{t_i=y_i}^{x_i-1} \Delta_i f(y_1, \dots, y_{i-1}, t_i, x_{i+1}, \dots, x_n) \right\} \\ + f(x) \sum_{i=1}^n \left\{ \sum_{t_i=y_i}^{x_i-1} \Delta_i g(y_1, \dots, y_{i-1}, t_i, x_{i+1}, \dots, x_n) \right\}.$$



New Weighted Multivariate
Grüss Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

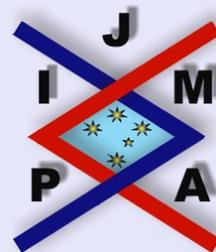
Page 14 of 18

Multiplying both sides of (4.3) by $w(y)$ and summing both sides of the resulting identity with respect to y over B , we have

$$\begin{aligned}
 (4.4) \quad & 2 \sum_y w(y) f(x) g(x) - g(x) \sum_y w(y) f(y) - f(x) \sum_y w(y) g(y) \\
 &= g(x) \sum_y \left(\sum_{i=1}^n \left\{ \sum_{t_i=y_i}^{x_i-1} \Delta_i f(y_1, \dots, y_{i-1}, t_i, x_{i+1}, \dots, x_n) \right\} \right) w(y) \\
 &+ f(x) \sum_y \left(\sum_{i=1}^n \left\{ \sum_{t_i=y_i}^{x_i-1} \Delta_i g(y_1, \dots, y_{i-1}, t_i, x_{i+1}, \dots, x_n) \right\} \right) w(y).
 \end{aligned}$$

Now, multiplying both sides of (4.4) by $w(x)$ and summing the resulting identity with respect to x on B we have

$$\begin{aligned}
 (4.5) \quad & 2 \left(\sum_y w(y) \right) \sum_x w(x) f(x) g(x) \\
 &- \left(\sum_x w(x) g(x) \right) \left(\sum_y w(y) f(y) \right) - \left(\sum_x w(x) f(x) \right) \left(\sum_y w(y) g(y) \right) \\
 &= \sum_x w(x) g(x) \left[\sum_y \left(\sum_{i=1}^n \left\{ \sum_{t_i=y_i}^{x_i-1} \Delta_i f(y_1, \dots, y_{i-1}, t_i, x_{i+1}, \dots, x_n) \right\} \right) w(y) \right] \\
 &+ \sum_x w(x) f(x) \left[\sum_y \left(\sum_{i=1}^n \left\{ \sum_{t_i=y_i}^{x_i-1} \Delta_i g(y_1, \dots, y_{i-1}, t_i, x_{i+1}, \dots, x_n) \right\} \right) w(y) \right].
 \end{aligned}$$



**New Weighted Multivariate
Grüss Type Inequalities**

B.G. Pachpatte

Title Page

Contents



Go Back

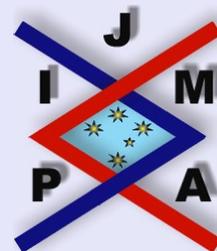
Close

Quit

Page 15 of 18

From (4.5) and using the properties of modulus we have

$$\begin{aligned}
 & |L(w, f, g)| \\
 & \leq \frac{1}{2 \sum_x w(x)} \left[\sum_x w(x) |g(x)| \right. \\
 & \quad \times \sum_y \left(\sum_{i=1}^n \left| \left\{ \sum_{t_i=y_i}^{x_i-1} |\Delta_i f(y_1, \dots, y_{i-1}, t_i, x_{i+1}, \dots, x_n)| \right\} \right| \right) w(y) \\
 & \quad + \sum_x w(x) |f(x)| \\
 & \quad \times \sum_y \left(\sum_{i=1}^n \left| \left\{ \sum_{t_i=y_i}^{x_i-1} |\Delta_i g(y_1, \dots, y_{i-1}, t_i, x_{i+1}, \dots, x_n)| \right\} \right| \right) w(y) \Big] \\
 & \leq \frac{1}{2 \sum_x w(x)} \sum_x w(x) \left[|g(x)| \sum_y \left(\sum_{i=1}^n \left| \left\{ \|\Delta_i f\|_\infty \sum_{t_i=y_i}^{x_i-1} 1 \right\} \right| \right) w(y) \right. \\
 & \quad \left. + |f(x)| \sum_y \left(\sum_{i=1}^n \|\Delta_i g\|_\infty |x_i - y_i| \right) w(y) \right] \\
 & = \frac{1}{2 \sum_x w(x)} \sum_x w(x) \left[\sum_x w(x) |g(x)| \sum_y \left(\sum_{i=1}^n \|\Delta_i f\|_\infty |x_i - y_i| \right) w(y) \right. \\
 & \quad \left. + |f(x)| \sum_y \left(\sum_{i=1}^n \|\Delta_i g\|_\infty |x_i - y_i| \right) w(y) \right]
 \end{aligned}$$



**New Weighted Multivariate
Grüss Type Inequalities**

B.G. Pachpatte

Title Page

Contents

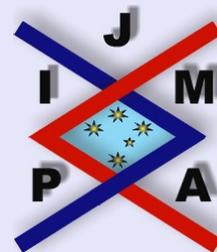


Go Back

Close

Quit

Page 16 of 18



Title Page

Contents



Go Back

Close

Quit

Page 17 of 18

$$= \frac{1}{2 \sum_x w(x)} \sum_x w(x) \left[|g(x)| \sum_y M[f, x_i, y_i] w(y) \right. \\ \left. + |f(x)| \sum_y M[g, x_i, y_i] w(y) \right],$$

which is the required inequality in (2.3).

The proof of the inequality (2.4) can be completed by following the proof of (2.3) and closely looking at the proof of (2.2). Here we omit the details. \square

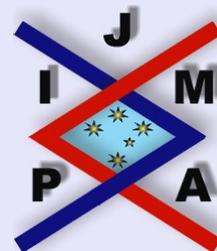
Remark 4.1. *Multiplying the left sides and right sides of (4.1) and (4.2), then multiplying the resulting identity by $w(y)$, summing it with respect to y over B , again multiplying the resulting identity by $w(x)$, summing it with respect to x over B and closely looking at the proof of the inequality (2.3) we get*

$$(4.6) \quad |L(w, f, g)| \\ \leq \frac{1}{2 \sum_x w(x)} \sum_x w(x) \left(\sum_y M[f, x_i, y_i] M[g, x_i, y_i] w(y) \right).$$

In concluding we note that in [2] Fink has given some Grüss type inequalities for measures other than the Lebesgue measure, including signed measures which provide different upper bounds. In addition, in [2] new proofs to some old results are also given. However, the inequalities established here are different and cannot be compared with those of given in [2].

References

- [1] S.S. DRAGOMIR, Some integral inequalities of Grüss type, *Indian J. Pure and Appl. Math.*, **31** (2000), 379–415.
- [2] A.M. FINK, A treatise on Grüss inequality, *Analytic and Geometric Inequalities and Applications*, T.M. Rassias and H.M. Srivastava (eds.), Kluwer Academic Publishers, Dordrecht 1999, 93–113.
- [3] G. GRÜSS, Über das maximum des absoluten Betrages von $\frac{1}{b-a} \int_a^b f(x) g(x) dx - \frac{1}{(b-a)^2} \int_a^b f(x) dx \int_a^b g(x) dx$, *Math. Z.*, **39** (1935), 215–226.
- [4] D.S. MITRINOVIĆ, J.E. PEČARIĆ AND A.M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.
- [5] B.G. PACHPATTE, On multidimensional Grüss type inequalities, *J. Inequal. Pure and Appl. Math.*, **3**(2) (2002), Art. 27. [ONLINE: <http://jipam.vu.edu.au>]
- [6] B.G. PACHPATTE, On multidimensional Ostrowski and Grüss type finite difference inequalities, *J. Inequal. Pure and Appl. Math.*, **4**(2) (2003), Art. 7. [ONLINE: <http://jipam.vu.edu.au>]
- [7] W. RUDIN, *Principles of Mathematical Analysis*, McGraw-Hill Book Company, Inc. 1953.



New Weighted Multivariate Grüss Type Inequalities

B.G. Pachpatte

Title Page

Contents



Go Back

Close

Quit

Page 18 of 18