journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 8 (2007), Issue 3, Article 75, 6 pp.

ON HADAMARD INTEGRAL INEQUALITIES INVOLVING TWO LOG-PREINVEX FUNCTIONS

MUHAMMAD ASLAM NOOR

MATHEMATICS DEPARTMENT
COMSATS INSTITUTE OF INFORMATION TECHNOLOGY
ISLAMABAD, PAKISTAN.
noormaslam@hotmail.com

Received 10 April, 2007; accepted 09 September, 2007
Communicated by B.G. Pachpatte

ABSTRACT. In this paper, we establish some new Hermite-Hadamard type integral inequalities involving two log-preinvex functions. Note that log-preinvex functions are nonconvex functions and include the log-convex functions as special cases. As special cases, we obtain the well known results for the convex functions.

Key words and phrases: Preinvex functions, Hermite-Hadamard integral inequalities, log-prinvex functions.

2000 *Mathematics Subject Classification.* 26D15, 26D99.

1. INTRODUCTION

In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of invex functions introduced by Hanson [4]. Hanson's initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Weir and Mond [14] and Noor [6, 7] have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems. It is well-known that the preinvex functions and invex sets may not be convex functions and convex sets. In recent years, several refinements of the Hermite-Hadamard inequalities have been obtained for the convex functions and its variant forms. In this direction, Noor [8] has established some Hermite-Hadamard type inequalities for preinvex and log-preinvex functions. In this paper, we establish some Hermite-Hadamard type inequalities involving two log-preinvex functions using essentially the technique of Pachpatte [11, 12]. This is the main motivation of this paper.

This research is supported by the Higher Education Commission, Pakistan, through research grant No: 1-28/HEC/HRD/2005/90.
111-07

2. PRELIMINARIES

Let K be a nonempty closed set in $\mathbb{R}^n$. We denote by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ the inner product and norm respectively. Let $f : K \to \mathbb{R}$ and $\eta(\cdot, \cdot) : K \times K \to \mathbb{R}$ be continuous functions. First of all, we recall the following well known results and concepts.

Definition 2.1 ([6, 14])**.** Let $u \in K$. Then the set K is said to be invex at u with respect to $\eta(\cdot, \cdot)$, if

$$u + t\eta(v, u) \in K, \quad \forall u, v \in K, \quad t \in [0, 1].$$

K is said to be an invex set with respect to η, if K is invex at each $u \in K$. The invex set K is also called a η-connected set.

Remark 2.1 ([1])**.** We would like to mention that Definition 2.1 of an invex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point u which is contained in K. We do not require that the point v should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that v should be an end point of the path for every pair of points $u, v \in K$, then $\eta(v, u) = v - u$, and consequently invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\eta(v, u) = v - u$, but the converse is not necessarily true, see [14, 15] and the references therein. For the sake of simplicity, we always assume that $K = [a, a + \eta(b, a)]$, unless otherwise specified.

Definition 2.2 ([14])**.** The function f on the invex set K is said to be preinvex with respect to η, if

$$f(u + t\eta(v, u)) \leq (1 - t)f(u) + tf(v), \quad \forall u, v \in K, \quad t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

Note that every convex function is a preinvex function, but the converse is not true. For example, the function $f(u) = -|u|$ is not a convex function, but it is a preinvex function with respect to η, where

$$\eta(v, u) = \begin{cases} v - u, & \text{if} \quad v \leq 0, u \leq 0 \quad \text{and} \quad v \geq 0, u \geq 0 \\ u - v, & \text{otherwise} \end{cases}$$

Definition 2.3. The differentiable function f on the invex set K is said to be an invex function with respect to $\eta(\cdot, \cdot)$, if

$$f(v) - f(u) \geq \langle f'(u), \eta(v, u) \rangle, \quad \forall u, v \in K,$$

where $f'(u)$ is the differential of f at u.

The concepts of the invex and preinvex functions have played very important roles in the development of generalized convex programming. From Definitions 2.2 and 2.3, it is clear that the differentiable preinvex functions are invex functions, but the converse is also true under certain conditions, see [5, 9, 10, 11].

Definition 2.4. The function f on the invex set K is called quasi preinvex with respect to $\eta(\cdot, \cdot)$, such that

$$f(u + t\eta(v, u)) \leq \max\{f(u), f(v)\}, \quad \forall u, v \in K, \quad t \in [0, 1].$$

Definition 2.5 ([6])**.** The function f on the invex set K is said to be logarithmic preinvex with respect to η, if

$$f(u + t\eta(v, u)) \leq (f(u))^{1-t}(f(v))^t, \quad u, v \in K, \quad t \in [0, 1],$$

where $F(\cdot) > 0$.

From the above definitions, we have

$$\begin{aligned} f(u+t\eta(v,u)) &\leq (f(u))^{1-t}(f(v))^t \\ &\leq (1-t)f(u)+tf(v) \\ &\leq \max\{f(u),f(v)\}. \end{aligned}$$

From Definition 2.5, we have

$$\log f(u+t\eta(v,u)) \leq (1-t)\log(f(u))+t\log(f(v)), \quad \forall u,v\in K, \quad t\in[0,1].$$

In view of this fact, we obtain the following.

Definition 2.6. The differentiable function f on the invex set K is said to be a log-invex function with respect to $\eta(\cdot,\cdot)$, if

$$\begin{aligned} \log f(v)-\log f(u) &\geq \left\langle \frac{d}{dt}(\log f((u)),\eta(v,u)\right\rangle \quad \forall u,v\in K \\ &= \left\langle \frac{f'(u)}{f(u)},\eta(v,u)\right\rangle. \end{aligned}$$

It can be shown that every differentiable log-preinvex function is a log-invex function, but the converse is not true. Note that for $\eta(v,u)=v-u$, the invex set K becomes the convex set and consequently the pre-invex, invex, and log-preinvex functions reduce to convex and log-convex functions.

It is well known [3, 12, 13] that if f is a convex function on the interval $I=[a,b]$ with $a<b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \quad \forall a,b\in I,$$

which is known as the Hermite-Hadamard inequality for the convex functions. For some results related to this classical result, see [2, 3, 12, 13] and the references therein. Dragomir and Mond [2] proved the following Hermite-Hadamard type inequalities for the log-convex functions:

$$\begin{aligned} f(\frac{a+b)}{2} &\leq \exp\left[\frac{1}{b-a}\int_a^b ln[f(x)]dx\right] \\ &\leq \frac{1}{b-a}\int_a^b G(f(x),f(a+b-x))dx \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq L(f(a),f(b)) \leq \frac{f(a)+f(b)}{2}, \end{aligned} \tag{2.1}$$

where $G(p,q)=\sqrt{pq}$ is the geometric mean and $L(p,q)=\frac{p-q}{\ln p-\ln q}\quad (p\neq q)$ is the logarithmic mean of the positive real numbers $p,q\quad$ (for $\quad p=q$, we put $L(p,q)=p$).

Pachpatte [11, 12] has also obtained some other refinements of the Hermite-Hadamard inequality for differentiable log-convex functions. In a recent paper, Noor [8] has obtained the following analogous Hermite-Hadamard inequalities for the preinvex and log-preinvex functions.

Theorem 2.2 ([8]). *Let $f:K=[a,a+\eta(b,a)]\longrightarrow(0,\infty)$ be a preinvex function on the interval of real numbers K° (the interior of I) and $a,b\in K^\circ$ with $a<a+\eta(b,a)$.Then the*

following inequality holds.

$$f\left(\frac{2a+\eta(b,a))}{2}\right) \leq \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Theorem 2.3 ([8]). *Let f be a log-preinvex function on the interval $[a, a+\eta(b,a)]$. Then*

$$\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x) \leq \frac{f(a)-f(b)}{\log f(a) - \log f(b)} = L\left(f(a), f(b)\right),$$

where $L(\cdot,\cdot)$ is the logarithmic mean.

The main purpose of this paper is to establish new inequalities involving two log-preinvex functions.

3. MAIN RESULTS

Theorem 3.1. *Let $f, g : K = [a, a+\eta(b,a)] \longrightarrow (0,\infty)$ be preinvex functions on the interval of real numbers K° (the interior of I) and $a, b \in K^\circ$ with $a < a+\eta(b,a)$. Then the following inequality holds.*

(3.1)
$$\frac{4}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)g(x)dx \leq \left[f(a)+f(b)\right] L\left(f(a), f(b)\right) + \left[g(a)+g(b)\right] L\left(g(a), g(b)\right).$$

Proof. Let f, g be preinvex functions. Then

$$f(a+t\eta(b,a)) \leq \left[f(a)\right]^{1-t}\left[f(b)\right]^t$$
$$g(a+t\eta(b,a)) \leq \left[g(a)\right]^{1-t}\left[g(b)\right]^t$$

Consider

$$\begin{aligned}
\int_a^{a+\eta(b,a)} f(x)g(x)dx &= \eta(b,a)\int_0^1 f\left(a+t\eta(b,a)\right) g\left(a+t\eta(b,a)\right) dt \\
&\leq \frac{\eta(b,a)}{2}\int_0^1 \left[\{f(a+t\eta(b,a))\}^2 + \{g(a+t\eta(b,a))\}^2\right] dt \\
&\leq \frac{\eta(b,a)}{2}\int_0^1 \left[\left[[f(a)]^{1-t}[f(b)]^t\right]^2 + \left[[g(a)]^{1-t}[g(b)]^t\right]^2\right] dt \\
&= \frac{\eta(b,a)}{2}\left\{[f(a)]^2\int_0^1\left[\frac{f(b)}{f(a)}\right]^{2t} dt + [g(a)]^2\int_0^1\left[\frac{g(b)}{g(a)}\right]^{2t} dt\right\} \\
&= \frac{\eta(b,a)}{4}\left\{[f(b)]^2\int_0^2\left[\frac{f(b)}{f(a)}\right]^{w} dw + [g(a)]^2\int_0^2\left[\frac{g(b)}{g(a)}\right]^{w} dw\right\} \\
&= \frac{\eta(b,a)}{4}\left\{[f(a)]^2\left[\frac{\left[\frac{f(b)}{f(a)}\right]^w}{\log\frac{f(b)}{f(a)}}\right]_0^2 + [g(a)]^2\left[\frac{\left[\frac{g(b)}{g(a)}\right]^2}{\log\frac{g(b)}{g(a)}}\right]_0^2\right\} \\
&= \frac{\eta(b,a)}{4}\left\{\frac{[f(a)+f(b)][f(b)-f(a)]}{\log f(b)-\log f(a)} + \frac{g(a)+g(b)][g(b)-g(a)]}{\log g(b)-\log g(a)}\right\} \\
&= \frac{\eta(b,a)}{4}\left\{\left[f(b)+f(a)\right]L(f(b),f(a)) + \left[g(b)+g(a)\right]L(g(b),g(a))\right\},
\end{aligned}$$

which is the required (3.1). This completes the proof. □

For the differentiable log-invex functions, we have the following result.

Theorem 3.2. *Let* $f, g : [a, a + \eta(b, a)] \longrightarrow (0, \infty)$ *be differentiable log-invex functions with* $a < a + \eta(b, a)$. *Then*

$$\begin{aligned} \frac{2}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)g(x)dx &\geq \frac{1}{\eta(b,a)} f\left(\frac{2a+\eta(b,a)}{2}\right) \int_a^{a+\eta(b,a)} g(x) \\ &\quad \times \exp\left[\left\langle \frac{f'\left(\frac{2a+\eta(b,a)}{2}\right)}{f\left(\frac{2a+\eta(b,a)}{2}\right)}, \eta\left(x, \frac{2a+\eta(b,a)}{2}\right)\right\rangle\right] dx \\ &\quad + \frac{1}{\eta(b,a)} g\left(\frac{2a+\eta(b,a)}{2}\right) \int_a^{a+\eta(b,a)} f(x) \\ &\quad \times \exp\left[\left\langle \frac{g'\left(\frac{2a+\eta(b,a)}{2}\right)}{g\left(\frac{2a+\eta(b,a)}{2}\right)}, \eta\left(x, \frac{2a+\eta(b,a)}{2}\right)\right\rangle\right] dx. \end{aligned} \tag{3.2}$$

Proof. Let f, g be differentiable log-invex functions. Then

$$\log f(x) - \log f(y) \geq \left\langle \frac{d}{dt}(\log f(y)), \eta(x,y) \right\rangle,$$

$$\log g(x) - \log g(y) \geq \left\langle \frac{d}{dt}(\log g(y)), \eta(x,y) \right\rangle, \quad \forall x, y \in K,$$

which implies that

$$\log \frac{f(x)}{f(y)} \geq \left\langle \frac{f'(y)}{f(y)}, \eta(x,y) \right\rangle.$$

That is,

$$f(x) \geq f(y) \exp\left[\langle \frac{f'(y)}{f(y)}, \eta(x,y) \rangle\right], \tag{3.3}$$

$$g(x) \geq g(y) \exp\left[\langle \frac{g'(y)}{g(y)}, \eta(x,y) \rangle\right]. \tag{3.4}$$

Multiplying both sides of (3.3) and (3.4) by $g(x)$ and $f(x)$ respectively, and adding the resultant, we have

$$2f(x)g(x) \geq g(x)f(x) \exp\left[\left\langle \frac{f'(y)}{f(y)}, \eta(x,y) \right\rangle\right] + f(x)g(x) \exp\left[\left\langle \frac{g'(y)}{g(y)}, \eta(x,y) \right\rangle\right]. \tag{3.5}$$

Taking $y = \frac{2a+\eta(b,a)}{2}$, in (3.5), we have

$$\begin{aligned} 2g(x)f(x) &\geq g(x) f\left(\frac{2a+\eta(b,a)}{2}\right) \exp\left[\left\langle \frac{f'\left(\frac{2a+\eta(b,a)}{2}\right)}{f\left(\frac{2a+\eta(b,a)}{2}\right)}, \eta\left(x, \frac{2a+\eta(b,a)}{2}\right)\right\rangle\right] \\ &\quad + f(x) g\left(\frac{2a+\eta(b,a)}{2}\right) \exp\left[\left\langle \frac{g'\left(\frac{2a+\eta(b,a)}{2}\right)}{g\left(\frac{2a+\eta(b,a)}{2}\right)}, \eta\left(x, \frac{2a+\eta(b,a)}{2}\right)\right\rangle\right], \\ &\qquad x \in [a, a+\eta(b,a)]. \end{aligned}$$

Integrating the above inequality with respect to x on $[a, a + \eta(b, a)]$, and dividing both sides of the resultant inequality by $\eta(b, a)$, we can obtain the desired inequality (3.2). □

REFERENCES

[1] T. ANTCZAK, Mean value in invexity analysis, *Nonl. Anal.*, **60** (2005), 1473–1484.

[2] S.S. DRAGOMIR AND B. MOND, Integral inequalities of Hadamard type for log-convex functions, *Demonst. Math.*, **31** (1998), 354–364.

[3] S.S. DRAGOMIR AND C.E.M. PEARCE, *Selected Topics on Hermite-Hadamard Type Inequalities*, RGMIA Monograph, Victoria University, Australia, 2000.

[4] M.A. HANSON, On sufficiency of the Kuhn-Tucker conditions, *J. Math. Anal. Appl.*, **80** (1981), 545–550.

[5] S.R. MOHAN AND S.K. NEOGY, On invex sets and preinvex functions, *J. Math. Anal. Appl.*, **189** (1995), 901–908.

[6] M. ASLAM NOOR, Variational-like inequalities, *Optimization*, **30** (1994), 323–330.

[7] M. ASLAM NOOR, Invex equilibrium problems, *J. Math. Anal. Appl.*, **302** (2005), 463–475.

[8] M. ASLAM NOOR, Hermite-Hadamard integral inequalities for log-preinvex functions, Preprint, 2007.

[9] M. ASLAM NOOR AND K. INAYAT NOOR, Some characterizations of strongly preinvex functions, *J. Math. Anal. Appl.*, **316** (2006), 697–706.

[10] M. ASLAM NOOR AND K. INAYAT NOOR, Hemiequilibrium problems, *Nonlinear Anal.*, **64** (2006), 2631–2642.

[11] B.G. PACHPATTE, A note on integral inequalities involving two log-convex functions, *Math. Inequal. Appl.*, **7** (2004), 511–515.

[12] B.G. PACHPATTE, *Mathematical Inequalities*, North-Holland Library, Vol. 67, Elsevier Science, Amsterdam, Holland, 2005.

[13] J.E. PEČARIĆ, F. PROSCHAN AND Y.L. TONG, *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.

[14] T. WEIR AND B. MOND, Preinvex functions in multiobjective optimization, *J. Math. Anal. Appl.*, **136** (1988), 29–38.

[15] X.M. YANG, X.Q. YANG AND K.L. TEO, Generalized invexity and generalized invariant monotonicity, *J. Optim. Theory Appl.*, **117** (2003), 607–625.