Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 5, Issue 4, Article 104, 2004

BOUNDS ON THE EXPECTATIONS OF k^{th} RECORD INCREMENTS

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Received 09 October, 2003; accepted 11 November, 2004
Communicated by C.E.M. Pearce

ABSTRACT. Here in this paper, we establish sharp bounds on the expectations of k^{th} record increments from general and non-negative parent distributions. We also determine the probability distributions for which the bounds are attained. The bounds are numerically evaluated and compared with other rough bounds.

Key words and phrases: Record statistics; Record increments; Bounds for moments; Moriguti monotone approximation.

2000 *Mathematics Subject Classification.* 62G30, 62G32, 62E10, 62E15.

1. INTRODUCTION

Consider independent identically (iid) distributed random variables $X_1, \dots, X_n, \dots,$ with a continuous common distribution function (cdf) F. We assume the parent cdf has finite mean $\mu = \int_0^1 F^{-1}(x)dx$ and finite variance $\sigma^2 = \int_0^1 (F^{-1}(x) - \mu)^2 dx$. The j^{th} order statistic $X_{j:n}$, $1 \le j \le n$, is the j^{th} smallest value in the finite sequence $X_1, X_2, \dots, X_n$. An observation X_j will be called an upper record statistic if its value exceeds that of all previous observations. That is, X_j is a record if $X_j > X_i$ for every $i < j$. The indices at which the records occur are called record times. The record times $T_n, n \ge 0$ can be defined as follows:

$$T_0 = 1,$$

and

$$T_n = \min\{j : j > T_{n-1} : \ X_j > X_{T_{n-1}}\}, \quad n \ge 1.$$

Then the sequence of record statistics $\{R_n\}$ is defined by

$$R_n = X_{T_n:T_n}, n = 0, 1, 2, \dots \ .$$

ISSN (electronic): 1443-5756

The problem of the paper was suggested by N. Papadatos during the stimulating research group meeting at the Banach Center, Polish Academy of Sciences, Warsaw in May 2002. The author would like to thank the University of Jordan for supporting this research work. Thanks are also due to the associate editor and referee for their comments and useful suggestions.

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By definition R_0 is a record statistic (trivial record).

Like extreme order statistics, record statistics are applied in estimating strength of materials, predicting natural disasters, sport achievements etc. Record statistics are closely connected with the occurrence times of some corresponding non-homogeneous Poisson processes often used in shock models (cf. Gupta and Kirmani, 1988). Record statistics are also used in reliability theory. Serious difficulties for the statistical inference based on records arise due to the fact that $ET_n = +\infty$, $n = 1, 2, \ldots$, and the occurrences of records are very rare in practice. These problems are removed once we consider the model of k^{th} record statistics proposed by Dziubdziela and Kopociński (1976).

For a positive integer k, let $T_{0,k} = k$ and

$$T_{n,k} = \min\{j : j > T_{n-1,k},\ X_j > X_{T_{n-1,k}-k+1:T_{n-1,k}}\}, \qquad n \geq 1.$$

Then $R_{n,k} = X_{T_{n,k}-k+1:T_{n,k}}$, and $T_{n,k},\ n \geq 0$, are the sequences of k^{th} record statistics and k^{th} record times, respectively. Obviously, we obtain ordinary record statistics in the case of $k = 1$. In reliability theory, the n^{th} value of k^{th} record statistics is just the failure time of a $k-$out-of-$T_{n,k}$ system. For more details about record statistics, and their distributional properties, one may refer to Ahsanullah (1995), Arnold et al. (1998) and Ahsanullah and Nevzorov (2001).

Several researchers have discussed the subject of moment bounds of order statistics. Moriguti (1953) suggested sharp bounds for the expectations of single order statistics based on a monotone approximation of respective density functions of standard uniform samples by means of the derivatives of the greatest convex minorants of their antiderivatives. Simple analytic formulae for the sample maxima were given in Gumbel (1954), and Hartley and David (1954). Arnold (1985) presented more general sharp bounds for the maximum and arbitrary combination of order statistics, respectively, of possibly dependent samples in terms of central absolute moments of various orders based on the Hölder inequality. Papadatos (1997) established exact bounds for the expectations of order statistics from non-negative populations.

In the context of record statistics, Nagaraja (1978) presented analytic formulae for the sharp bounds of the ordinary records, based on application of the Schwarz inequality. By the same approach, Grudzień and Szynal (1985) obtained nonsharp bounds for k^{th} record statistics. Raqab (1997) improved the results using a greatest convex minorant approach. Raqab (2000) evaluated bounds on expectations of ordinary record statistics based on the Hölder inequality. Gajek and Okolewski (1997) applied the Steffensen inequality to derive different bounds on expectations of order and record statistics.

Recently, Raqab and Rychlik (2002) presented sharp bounds for the expectations of k^{th} record statistics in various scale units for a general distribution.

Generally, for $1 \leq m < n$, we have

$$E(R_{n,k} - R_{m,k}) = \int_0^1 [F^{-1}(x) - \mu] h_{m,n,k}(x)dx, \qquad 1 \leq m < n, \tag{1.1}$$

where

$$h_{m,n,k}(x) = f_{n,k}(x) - f_{m,k}(x), \qquad 0 \leq x \leq 1,$$

and

$$f_{n,k}(x) = \frac{k^{n+1}}{n!}[-\ln(1-x)]^n (1-x)^{k-1}, \qquad k \geq 1,\ n \geq 0,$$

is the density function of the n^{th} value of the k^{th} records of the iid standard uniform sequence (cf., e.g., Arnold et al., 1998, p. 81). For simplification, we change the variables and obtain another representation of (1.1),

$$E(R_{n,k} - R_{m,k}) = \int_0^\infty F^{-1}(1 - e^{-y})\varphi_{m,n,k}(y)e^{-y}dy, \tag{1.2}$$

where

$$\varphi_{m,n,k}(y) = g_{n,k}(y) - g_{m,k}(y),$$

and

$$g_{n,k}(y) = \frac{k^{n+1}}{n!} y^n e^{-(k-1)y}, \qquad y > 0,$$

is a density function with respect to the exponential measure on the positive half-axis. The respective antiderivative is

$$\Phi_{m,n,k}(y) = G_{n,k}(y) - G_{m,k}(y) = IG_y(n+1,k) - IG_y(m+1,k),$$

where $IG_x(a,b)$ stands for the incomplete gamma function. This antiderivative can be rewritten in the following form:

$$\Phi_{m,n,k}(y) = -e^{-ky} \sum_{j=m+1}^{n} \frac{(ky)^j}{j!}. \tag{1.3}$$

Applying the Cauchy-Schwarz inequality to (1.2), we obtain a classical nonsharp bound of $E(R_{n,k} - R_{m,k})$

$$E(R_{n,k} - R_{m,k}) \le B_{m,n,k}(1)\sigma,$$

where

$$B_{m,n,k}(1) = \left\{ k \left(\frac{k}{2k-1}\right)^{2m+1} \binom{2m}{m} + k \left(\frac{k}{2k-1}\right)^{2n+1} \binom{2n}{n} -2k \left(\frac{k}{2k-1}\right)^{m+n+1} \binom{m+n}{m} \right\}^{\frac{1}{2}}. \tag{1.4}$$

In Section 2 of this paper, we establish sharp bounds for the expectations of k^{th} record increments expressed in terms of scale units σ. In Section 3, we establish bounds for the moments of k^{th} record increments for non-negative parent populations. Computations and comparisons between the classical bounds and the ones derived in Sections 2 and 3 are presented and discussed in Section 4.

2. BOUNDS ON EXPECTATIONS OF k^{th} RECORD INCREMENTS

In this section we present projection moment bounds on the expectations of k^{th} record increments in terms of scale units. First we recall Moriguti's (1953) approach that will be used in this section. Suppose that a function h has a finite integral on $[a,b]$. Let $H(x) = \int_a^x h(t)dt,\ a \le x \le b$, stand for its antiderivative, and $\overline{H}$ be the greatest convex minorant of H. Further, let $\overline{h}$ be a nondecreasing version of the derivative (e.g. right continuous) of $\overline{H}$. Obviously, $\overline{h}$ is a nondecreasing function and constant in the interval where $\overline{h} \neq h$. For every nondecreasing function w on $[a,b]$ for which both the integrals in (2.1) are finite, we have

$$\int_a^b w(x)h(x)dx \le \int_a^b w(x)\bar{h}(x)dx. \tag{2.1}$$

The equality in (2.1) holds iff w is constant in every interval contained in the set, where $\bar{H} \neq H$.

Analyzing the variability of $h_{m,n,k}(x)$ is necessary for evaluations of optimal bounds. We consider first the problem with $m = n-1$ $(n \ge 2)$ and $k > 1$. For simplicity, we use $h_{n,k}(x), \varphi_{n,k}(x)$, and $B_{n,k}(i)$; $i = 1,2,3$ instead of $h_{n-1,n,k}(x)$, $\varphi_{n-1,n,k}(x)$, and $B_{n-1,n,k}(i)$; $i = 1,2,3$.

Function $h_{n,k}(x)$ can be represented as

$$h_{n,k}(x) = -f_{n-1,k}(x)\left[\frac{k}{n}\ln(1-x)+1\right], \quad n \geq 2.$$

It starts from the origin and vanishes as x approaches 1 passing the horizontal axis at $x = 1 - e^{-n/k}$ $(n \geq 2,\ k > 1)$. By using the facts that

$$f_{n,k}(x) = \frac{k}{n}[-\ln(1-x)]f_{n-1,k}(x) \quad \text{and}$$
$$f'_{n,k}(x) = \frac{k}{n}\left[n + (k-1)\ln(1-x)\right](1-x)^{-1}f_{n-1,k}(x),$$

we conclude that

$$h'_{n,k}(x) = -\frac{k}{n-1}f_{n-2,k}(x)(1-x)^{-1} \times \left\{\frac{k(k-1)}{n}[-\ln(1-x)]^2 + (2k-1)\ln(1-x) + (n-1)\right\}. \tag{2.2}$$

It follows from (2.2) that $h_{n,k}(x)$ decreases on $(0, a_{n,k})$, $(b_{n,k}, 1)$ and increases on $(a_{n,k}, b_{n,k})$, where $a_{n,k} = 1 - e^{-c_{n,k}}$, $b_{n,k} = 1 - e^{-d_{n,k}}$ with

$$c_{n,k} = \frac{(2k-1)n - \sqrt{(2k-1)^2 n + n(n-1)}}{2k(k-1)},$$

$$d_{n,k} = \frac{(2k-1)n + \sqrt{(2k-1)^2 n + n(n-1)}}{2k(k-1)}.$$

We can easily check that $h_{n,k}(a_{n,k}) < 0$ and $h_{n,k}(b_{n,k}) > 0$.

The antiderivative $H_{n,k}(x)$ of $h_{n,k}(x)$, needed for the Moriguti projection, is therefore concave decreasing, convex decreasing, convex increasing and concave increasing in $[0, a_{n,k}]$, $[a_{n,k}, 1-e^{-n/k}]$, $[1-e^{-n/k}, b_{n,k}]$, $[b_{n,k}, 1]$, respectively. Further, it is negative with $H_{n,k}(0) = H_{n,k}(1) = 0$. Thus its greatest convex minorant $\overline{H}_{n,k}$ is linear in $[0, 1-e^{-\beta}]$, and $[1-e^{-n/(k-1)}, 1]$ for some $\beta \in [c_{n,k}, n/k]$. That is,

$$\overline{H}_{n,k}(x) = \begin{cases} h_{n,k}(1-e^{-\beta})x, & \text{if } \ x \leq 1-e^{-\beta}, \\ H_{n,k}(x), & \text{if } \ 1-e^{-\beta} < x < 1-e^{-n/(k-1)}, \\ -h_{n,k}(1-e^{-n/(k-1)})(1-x), & \text{if } \ 1-e^{-n/(k-1)} \leq x \leq 1, \end{cases}$$

where β is determined numerically by the equation

$$\Phi_{n,k}(y) = \varphi_{n,k}(y)(1-e^{-y}). \tag{2.3}$$

Note that $y = n/(k-1)$ is obtained by solving the equation

$$\Phi_{n,k}(y) = -\varphi_{n,k}(y)e^{-y}. \tag{2.4}$$

The projection of $\varphi_{n,k}(y)$ onto the convex cone of nondecreasing functions in $L^2([0,\infty), e^{-y}\,dy)$ (cf. Rychlik, 2001, pp. 14-16) is

$$\overline{\varphi}_{n,k}(y) = \begin{cases} \varphi_{n,k}(\beta), & \text{if } \ y \leq \beta, \\ \varphi_{n,k}(y), & \text{if } \ \beta < y < \frac{n}{k-1}, \\ \varphi_{n,k}(\frac{n}{k-1}), & \text{if } \ y \geq \frac{n}{k-1}. \end{cases} \tag{2.5}$$

By (1.2), (2.5), and the Cauchy-Schwarz inequality, we get

$$(2.6)\quad \begin{aligned} E(R_{n,k}-R_{n-1,k}) &= \int_0^\infty [F^{-1}(1-e^{-y})-\mu][\varphi_{n,k}(y)-c]e^{-y}dy \\ &\le \int_0^\infty [F^{-1}(1-e^{-y})-\mu][\overline{\varphi}_{n,k}(y)-c]e^{-y}dy \\ &\le \left\{\int_0^\infty [\overline{\varphi}_{n,k}(y)-c]^2 e^{-y}dy\right\}^{\frac{1}{2}}\sigma, \end{aligned}$$

for arbitrary real c. The former inequality becomes equality if $F^{-1}(1-e^{-y})-\mu$ is constant on $(0,\beta)$ and $(n/(k-1),\infty)$. The latter one is attained if

$$(2.7)\quad F^{-1}(1-e^{-y})-\mu = \alpha|\overline{\varphi}_{n,k}(y)-c|\operatorname{sgn}(\overline{\varphi}_{n,k}(y)-c),\quad \alpha\ge 0.$$

The condition in (2.7) implies the former condition. As a consequence of that, the bound in (2.6) is attained for arbitrary c by the distribution function satisfying (2.7). Now we minimize the bound in the RHS of (2.6) with respect to $c=\varphi_{n,k}(\eta),\ \eta\in(\beta,n/(k-1))$. We have

$$(2.8)\quad \begin{aligned} &\int_0^\infty (\overline{\varphi}_{n,k}(y)-\varphi_{n,k}(\eta))^2 e^{-y}dy \\ &= [\varphi_{n,k}(\eta)-\varphi_{n,k}(\alpha)]^2(1-e^{-\beta}) + \int_\beta^\eta [\varphi_{n,k}(\eta)-\varphi_{n,k}(y)]^2 e^{-y}dy \\ &\quad + \int_\eta^{n/(k-1)} [\varphi_{n,k}(y)-\varphi_{n,k}(\eta)]^2 e^{-y}dy + [\varphi_{n,k}(n/(k-1))-\varphi_{n,k}(\eta)]^2 e^{-n/(k-1)}. \end{aligned}$$

Differentiation of the RHS of (2.8) and equating the result to 0 leads to $\varphi_{n,k}(\eta)=0$. This shows that the unique solution of (2.8) is $\eta=\eta^*=n/k$. It follows that the optimal bound on $E(R_{n,k}-R_{n-1,k})$ is given by

$$(2.9)\quad B_{n,k}(2) = \left\{\int_0^\infty [\overline{\varphi}_{n,k}(y)]^2 e^{-y}dy\right\}^{\frac{1}{2}}.$$

Summing up, (2.9) with (2.3) and (2.4) leads to the following bound

$$(2.10)\quad \begin{aligned} B_{n,k}(2) = \Bigg\{ &\varphi_{n,k}^2(\beta)(1-e^{-\beta}) + \varphi_{n,k}^2\left(\frac{n}{k-1}\right)e^{-\frac{n}{k-1}} \\ &+ \frac{k^{2n+2}}{(2k-1)^{2n+1}}\binom{2n}{n}\delta\left(2n+1,\frac{1}{2k-1}\right) \\ &+ \frac{k^{2n}}{(2k-1)^{2n-1}}\binom{2n-2}{n-1}\delta\left(2n-1,\frac{1}{2k-1}\right) \\ &- 2\frac{k^{2n+1}}{(2k-1)^{2n}}\binom{2n-1}{n-1}\delta\left(2n,\frac{1}{2k-1}\right)\Bigg\}^{\frac{1}{2}}, \end{aligned}$$

where $\delta(i,j)=IG_{n/(k-1)}(i,j)-IG_\beta(i,j)$, and β is the unique solution to

$$(2.11)\quad [(k-1)y-n]e^{-y} = ky-n,\quad n\ge 2,\ k>1.$$

From (2.7), the optimal bound is attained iff

$$(2.12)\quad F^{-1}(1-e^{-y})-\mu = \alpha|\overline{\varphi}_{n,k}(y)|\operatorname{sgn}(\overline{\varphi}_{n,k}(y)).$$

Note that the right-hand side of (2.12) is non-decreasing, negative on $(0, n/k)$ and positive on $(n/k, \infty)$. Moreover, this is constant on $(0, \beta)$ and $(n/(k-1), \infty)$, which is necessary and sufficient for the equality in the former inequality of (2.6). The condition

$$\int_0^\infty [F^{-1}(1-e^{-y}) - \mu]^2 e^{-y} dy = \sigma^2$$

forces $\alpha = \sigma / B_{n,k}(2)$. Consequently, the distributions functions of the location-scale family for which the bounds are attained have the form

$$F(x) = \begin{cases} 0, & \text{if } \; x \le \xi_1, \\ h^{-1}_{n-1,n,k}(B_{n,k}(2)\frac{x-\mu}{\sigma}), & \text{if } \; \xi_1 < x < \xi_2, \\ 1, & \text{if } \; x \ge \xi_2, \end{cases} \tag{2.13}$$

where

$$\xi_1 = \mu - \frac{\sigma}{B_{n,k}(2)} \varphi_{n,k}(\beta),$$

and

$$\xi_2 = \mu + \frac{\sigma}{B_{n,k}(2)} \varphi_{n,k}\left(\frac{n}{k-1}\right).$$

The distribution function in (2.13) is involving the inverse of smooth component $h_{n-1,n,k}$ with two atoms of measures $1 - e^{-\beta}$ and $e^{-n/(k-1)}$, respectively, at the ends of support.

Remark 2.1. In the special case of ordinary records ($m = n-1, k = 1$), Eq. (2.11) reduces to $n(1-e^{-y}) = y$ and the optimal bound coincides with the corresponding bound in Rychlik (2001, pp.141). The optimal bound for the extreme case $n = 1$ cannot be obtained from the above bound. Further, the case $n = k = 1$, leads to the estimates for $E(R_{1,1} - R_{0,1}) = E(R_{1,1} - \mu)$ which were already presented in Raqab and Rychlik (2002).

Now we consider the case $n = 1$ and $k > 1$. In this case, the projection of $h_{1,k}(x)$ onto the family of nondecreasing functions in the Hilbert space $L^2([0,1], dx)$ is $\overline{h}_{1,k}(x) = h_{1,k}(\min\{x, 1 - e^{-1/k-1}\})$.

From (2.1), we get

$$\begin{aligned} E(R_{1,k} - X_{1:k}) &\le \int_0^\infty \left[F^{-1}(1-e^{-y}) - \mu\right] \overline{\varphi}_{1,k}(y) e^{-y} dy \\ &\le B_{1,k}(2)\sigma, \end{aligned}$$

where

$$\begin{aligned} B_{1,k}(2) = \Bigg\{ &\frac{k^2 e^{-2}}{(k-1)^2} e^{-\frac{1}{k-1}} + \frac{k^2}{(2k-1)^3}(2k^2 - 2k + 1) \\ &- \frac{k^2 e^{-\frac{2k-1}{k-1}}}{(2k-1)^3 (k-1)^2}(6k^4 - 4k^3 + k^2 - 2k + 1) \Bigg\}^{\frac{1}{2}}. \end{aligned}$$

Using similar arguments to those in the previous proof, we conclude that the bound $B_{1,k}(2)$ is attained for the distribution function of the location-scale family

$$F(x) = \begin{cases} 0, & \text{if } \; x \le \mu - \frac{\sigma}{B_{1,k}(2)} k, \\ h^{-1}_{0,1,k}(B_{1,k}(2)\frac{x-\mu}{\sigma}), & \text{if } \; \mu - \frac{\sigma}{B_{1,k}(2)} k < x < \mu + \frac{\sigma}{B_{1,k}(2)} \cdot \frac{ke^{-1}}{k-1}, \\ 1, & \text{if } \; x \ge \mu + \frac{\sigma}{B_{1,k}(2)} \cdot \frac{ke^{-1}}{k-1}, \end{cases} \tag{2.14}$$

The distribution function in (2.14) has a jump of height $e^{-1/(k-1)}$ at the right end of support.

In the case of ordinary records ($k = 1$), one can establish optimal moment bounds for general k^{th} record increments $R_{n,1} - R_{m,1}$, $1 \le m < n$. The function $\varphi_{m,n}(y) = g_{n,1}(y) - g_{m,1}(y)$ can be rewritten as

$$\varphi_{m,n}(y) = g_{m,1}(y)\left[\frac{m!}{n!}y^{n-m} - 1\right], \quad 1 \le m < n.$$

We can easily note that function $h_{m,n}(x) = \varphi_{m,n}(-\ln(1-x))$ starts from the origin, decreases to $\varphi_{m,n}(1-e^{-\nu}) < 0$, where $\nu = [(n-1)!/(m-1)!]^{1/(n-m)}$ and then increases to ∞ at 1 passing the horizontal axis at $1 - e^{-\nu^*}$, where $\nu^* = [n!/m!]^{1/(n-m)}$. The antiderivative $H_{m,n}(x)$ needed in making the projection, is then concave decreasing, convex decreasing, and convex increasing in $[0, 1-e^{-\nu}]$, $[1-e^{-\nu}, 1-e^{-\nu^*}]$, and $[1-e^{-\nu^*}, 1]$, respectively, with $H_{m,n}(0) = H_{m,n}(1) = 0$. The corresponding greatest convex minorant $\overline{H}_{m,n}(x)$ is linear in $[0, \beta^*]$ for some $\beta^* \in [1-e^{-\nu}, 1-e^{-\nu^*}]$, that is determined numerically by the following equation

$$\Phi_{m,n}(y) = \varphi_{m,n}(y)(1-e^{-y}). \tag{2.15}$$

By (1.3), Eq. (2.15) can be simplified as

$$e^{-y}\sum_{j=m+1}^{n}\frac{y^j}{j!} = \left(\frac{y^m}{m!} - \frac{y^n}{n!}\right)(1-e^{-y}). \tag{2.16}$$

Finally the projection of $\varphi_{m,n}(y)$ in $L([0,\infty), e^{-y}dy)$ is

$$\overline{\varphi}_{m,n}(y) = \varphi_{m,n}\left(\max\{\beta^*, y\}\right).$$

Hence

$$\frac{E(R_{n,k} - R_{m,k})}{\sigma} \le \int_0^\infty [\overline{\varphi}_{m,n}(y) - c]^2 e^{-y}dy, \tag{2.17}$$

where $c = \varphi(\eta), \eta \in (\beta^*, \infty)$. The constant $\eta = \eta^* = \varphi^{-1}(1)$ minimizes the RHS of (2.17), and then the optimal bound simplifies to

$$B_{m,n}(2) = \varphi^2_{m,n}(\beta^*)(1-e^{-\beta^*}) - 1 + e^{-\beta^*}\left[\binom{2n}{n}\sum_{j=0}^{2n}\frac{\beta^{*j}}{j!} + \binom{2m}{m}\sum_{j=0}^{2m}\frac{\beta^{*j}}{j!} - 2\binom{m+n}{m}\sum_{j=0}^{m+n}\frac{\beta^{*j}}{j!}\right]^{\frac{1}{2}}. \tag{2.18}$$

The bound is attained by

$$F(x) = \varphi^{-1}_{m,n}\left(B_{m,n}(2)\frac{x-\mu}{\sigma}\right), \quad \mu - \sigma\frac{\varphi_{m,n}(\beta^*)}{B_{m,n}(2)} \le x < \infty. \tag{2.19}$$

The distribution (2.19) has a jump of height β^* and a density with infinite support to the right of the jump point.

3. BOUNDS FOR NON-NEGATIVE DISTRIBUTIONS

In this section, we develop bounds for the moments of k^{th} record increments from non-negative parent distributions. The bounds are expressed in terms of location units rather than scale units. The expectation of k^{th} record increments can be represented as

$$E(R_{n,k} - R_{m,k}) = \int_0^\infty [G_{m,k}(S(y)) - G_{n,k}(S(y))]dy, \tag{3.1}$$

where $S(y) = -ln(1 - F(y)),\ 0 < y < \infty$ is the hazard function.

In order to get optimal evaluations for the expectation in (3.1), we should analyze variablility of the following function:

$$W(y) = \frac{G_{m,k}(y) - G_{n,k}(y)}{e^{-y}}, \quad 0 \le y < \infty.$$

For $n = m+1$, it is clear to note that the function $W(y)$ is unimodal with mode $\gamma = \frac{m+1}{k-1}$. With $n > m+1$, a simple analysis leads to the conclusion that

$$W(y) = \sum_{j=m+1}^{n} q_j(y),$$

where

$$q_j(y) = \frac{(ky)^j e^{-(k-1)y}}{j!}.$$

Function $W'(y) > 0$ if $y \le \frac{m+1}{k-1}$ and $W'(y) < 0$ if $y \ge \frac{n}{k-1}$. By the continuity of $W(y)$, there exists a root of $W'(y)$, say $\gamma \in \left[\frac{m+1}{k-1}, \frac{n}{k-1}\right]$. The derivative of $W(y)$ can be written as

$$W'(y) = \frac{e^{-y}(g_{m,k}(y) - g_{n,k}(y)) + (G_{m,k}(y) - G_{n,k}(y))}{e^{-y}}.$$

Since $G'_{n,k}(y) = g_{n,k}(y)e^{-y}$, we have

$$[e^{-y}W'(y)]' = \frac{e^{-y}}{y}\left\{[m - (k-1)y]g_{m,k}(y) - [n - (k-1)y]g_{n,k}(y)\right\}.$$

We observe that the function $[e^{-y}W'(y)]' < 0$ for $y \in \left[\frac{m}{k-1}, \frac{n}{k-1}\right]$. This leads to the conclusion that $[e^{-y}W'(y)]$ is strictly decreasing and then the root $\gamma \in \left[\frac{m+1}{k-1}, \frac{n}{k-1}\right]$ must be unique. Consequently, $W(y)$ is unimodal function with mode γ. The value of γ can be evaluated numerically from the equation

$$G_{m,k}(y) - G_{n,k}(y) = (g_{n,k}(y) - g_{m,k}(y))\, e^{-y}. \tag{3.2}$$

For $m = n-1$, $\gamma = n/(k-1)$, which is the unique solution to (2.4).

From the non-negativity assumption, we have

$$\begin{aligned} E(R_{n,k} - R_{m,k}) &= \int_0^\infty W(S(y))(1 - F(y))dy \\ &\le (g_{n,k}(\gamma) - g_{m,k}(\gamma))\mu, \end{aligned} \tag{3.3}$$

which leads to

$$B_{m,n,k}(3) = \frac{k^{m+1}}{m!} e^{-(k-1)\gamma}\left[\frac{m!k^{n-m}}{n!}\gamma^{n-m} - 1\right]\mu, \tag{3.4}$$

where γ is the unique solution to

$$\sum_{j=m+1}^{n} \frac{k^j y^j}{j!} = \frac{k^{m+1}}{m!} y^m \left[\frac{k^{n-m}m!}{n!} y^{n-m} - 1\right]. \tag{3.5}$$

Note that Eq. (3.5) is a reduction of (3.2). The bound (3.3) is attained in the limit by a two-point marginal distribution supported at 0 and μe^{γ} with respective probabilities $1 - e^{-\gamma}$ and $e^{-\gamma}$.

For the special case $m = n-1$, $\gamma = \frac{n}{k-1},\ n \ge 2$ and the bound $B_{m,n,k}(3)$ can be simplified as

$$B_{n-1,n,k}(3) = \left(\frac{k}{k-1}\right)^n \frac{(n-1)^{n-1}}{(n-1)!} e^{-n}, \quad n \ge 2\ , k > 1.$$

A useful approximation for $n!$ where n is large, is given by Stirling's formula $n! \cong \sqrt{2\pi n}n^n e^{-n}$. This leads to a simpler formula

$$B_{n-1,n,k}(3) \cong \left(\frac{k}{k-1}\right)^n \cdot \frac{e^{-1}}{\sqrt{2\pi(n-1)}}.$$

4. COMPUTATIONS AND DISCUSSION

In this section we carry out a numerical study in order to compute the sharp bounds on the expectations of the n^{th} values of the k^{th} record increments for selected values of m, n and k. The first step of our calculations is to determine the parameters β, β^* and γ by solving equations (2.3), (2.15) and (3.2) whose left-hand side can be simplified and rewritten in terms of a Poisson sum of probabilities. Consequently, we numerically solve the equivalent equations (2.11), (2.16) and (3.5), respectively, by means of the Newton-Raphson method. Then using (2.10), (2.18), and (3.4), we evaluate the sharp bounds $B_{n,k}(2),\ B_{m,n}(2)\ (k = 1)$ and $B_{n,k}(3)$ for some selected values of m, n and k.

In Table 4.1, each optimal bound $B_{n,k}(2)$ is compared with the rough one $B_{n,k}(1)$ and the one for non-negative parent $B_{n,k}(3)$. Clearly, the rough bound results in a significant loss of accuracy in evaluating the k^{th} record increments. We observe that the bounds $B_{n,k}(2)$ and $B_{n,k}(3)$ decrease as k increases with fixed n which has the following explanation. If we consider μ and σ as general location and scale parameters and increase k, we restrict ourselves to narrower classes of distributions and the bounds in the narrower classes become tighter. Moreover, the relative discrepancy between $B_{n,k}(2)$ and $B_{n,k}(3)$ increases with the increase of parameter k. In fact, one can also argue that the bounds $B_{n,k}(1)$ strictly majorize $B_{n,k}(2)$ for $n \geq 1$ and $k > 1$. For this, the discrepancy between $B_{n,k}(1)$ and $B_{n,k}(3)$ is much larger than that between $B_{n,k}(2)$ and $B_{n,k}(3)$. For $n \geq k$, other calculations show that $B_{n,k}(1)$ and $B_{n,k}(2)$ beat $B_{n,k}(3)$.

Table 4.2 compares the rough bounds $B_{m,n}(1)$ with $B_{m,n}(2)$ for the moments of ordinary record increments ($k = 1; 1 \leq m < n$). The numerical results show that the application of the Hölder inequality combined with the Moriguti modification results in improvements in evaluating the moments bounds for record increments ($k = 1, 1 \leq m < n$). We have excluded $B_{m,n}(3)$ since it cannot be obtained for the ordinary records increments. Obviously, the bounds for non-negative distributions are expressed in terms of location units and these bounds beat the one derived based on combining the Moriguti approach with the Cauchy-Schwarz inequality when the coefficient of variation σ/μ exceeds the ratio $B_{n,k}(3)/B_{n,k}(2)$ depending on $n \geq 1$ and $k > 1$.

The aim of this paper was the development of the optimal moment bounds for the k^{th} record increments from both general and non-negative parent distributions. The results can be used effectively in estimating the expected values of records as well as in characterizing the probability distributions for which the bounds are attained. Possibly, one open problem is to find the sharp bounds in some restricted families of distributions, e.g. ones with symmetric distributions or with monotone failure rate.

Table 4.1: Bounds on the expectations of k^{th} records increments in various location or scale units.

n	k	β	$B_{n,k}(1)$	$B_{n,k}(2)$	$B_{n,k}(3)$
2	3	0.3948	0.6024	0.5681	0.6090
	4	0.2838	0.6293	0.5803	0.4812
	5	0.2213	0.6667	0.6053	0.4229
	6	0.1813	0.7063	0.6341	0.3898
3	4	0.5639	0.5182	0.4636	0.5311
	5	0.4409	0.5300	0.4633	0.4376
	6	0.3617	0.5492	0.4720	0.3871
	7	0.3065	0.5712	0.4846	0.3558
4	6	0.5410	0.4774	0.3994	0.4051
	7	0.4588	0.4891	0.4023	0.3619
	8	0.3982	0.5031	0.4084	0.3333
	9	0.3517	0.5183	0.4162	0.3129
5	7	0.6106	0.4432	0.3573	0.3793
	8	0.5302	0.4509	0.3576	0.3421
	9	0.4684	0.4607	0.3606	0.3162
	10	0.4195	0.4715	0.3651	0.2972
6	8	0.6618	0.4183	0.3266	0.3579
	9	0.5849	0.4238	0.3259	0.3256
	10	0.5239	0.4310	0.3271	0.3022
	11	0.4744	0.4391	0.3298	0.2846
10	14	0.6640	0.3646	0.2524	0.2625
	15	0.6185	0.3680	0.2524	0.2494
	16	0.5789	0.3719	0.2528	0.2386
	17	0.5440	0.3761	0.2537	0.2294

Table 4.2: Bounds on the expectations of ordinary records increments ($k = 1, 1 \leq m < n$) in various scale units.

m	n	β^*	$B_{m,n}(1)$	$B_{m,n}(2)$
1	2	1.59362	1.4142	0.9905
	3	2.1270	3.7417	3.5943
	4	2.6188	7.8740	7.7991
	5	3.0855	15.5563	15.5150
2	3	2.8214	2.4495	2.2254
	4	3.3308	6.7823	6.6925
	5	3.8117	14.6969	14.6462
	6	4.2740	29.5635	29.5321
3	4	3.9207	4.4721	4.3485
	5	4.4149	12.6491	12.5929
	6	4.8898	27.8568	27.8204
	7	5.3511	56.6745	56.6489
4	5	4.9651	8.3666	8.2966
	6	5.4526	23.9583	23.9208
	7	5.9261	53.3104	53.2820
	8	6.3890	109.3160	109.2930
5	6	5.9849	15.8745	15.8333
	7	6.4703	45.8258	45.7984
	8	6.9447	102.7030	102.6790
	9	7.4103	211.8210	211.7990

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