Journal of Inequalities in Pure and Applied Mathematics

http://jipam.vu.edu.au/

Volume 7, Issue 1, Article 19, 2006

ON BUSEMANN SURFACE AREA OF THE UNIT BALL IN MINKOWSKI SPACES

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Received 18 May, 2005; accepted 22 November, 2005
Communicated by S.S. Dragomir

ABSTRACT. For a given d-dimensional Minkowski space (finite dimensional Banach space) with unit ball B, one can define the concept of surface area in different ways when $d \geq 3$. There exist two well-known definitions of surface area: the Busemann definition and Holmes-Thompson definition of surface area. The purpose of this paper is to establish lower bounds for the surface area of the unit ball in a d-dimensional Minkowski space in case of Busemann's definition, when $d \geq 3$.

Key words and phrases: Busemann surface area, Cross-section measure, Isoperimetrix, Intersection body, Mixed volumes, Projection body.

2000 *Mathematics Subject Classification.* 52A40, 46B20.

1. INTRODUCTION

It was shown by Gołąb (see [11] for details of this theorem) that in a two-dimensional Minkowski space the surface area of the the unit ball lies between 6 and 8 where the extreme values are attained if and only if the unit ball is a regular hexagon and a parallelogram, respectively. Recall that in a two-dimensional Minkowski space the surface area is defined by the induced norm of this space. One can also raise the following question: "What are the extremal values of the surface area of the unit ball in a d-dimensional Minkowski space, when $d \geq 3$?" To answer this question, first the notion of surface area needs to be defined, since the norm is no longer sufficient to define the surface area, when $d \geq 3$. Various definitions of surface area were explored in higher dimensional Minkowski spaces (see [2, 3, 4, 12, 13]).

One of the definitions of surface area was given by Busemann in his papers [1, 2, 3]. In [4], Busemann and Petty investigated this *Busemann definition of surface area* for the unit ball when $d \geq 3$. They proved that if B is the unit ball of a d-dimensional Minkowski space $\mathbf{M}^d = (\mathbf{R}^d, \ || \cdot ||)$, then its Busemann surface area $\nu_B(\partial B)$ is at most $2d\epsilon_{d-1}$, and is equal to $2d\epsilon_{d-1}$ if and only if B is a parallelotope. Here ϵ_d stands for the volume of the standard d-dimensional

ISSN (electronic): 1443-5756

I would like to thank the referee for his comments and suggestions.
156-05

Euclidean unit ball. They also raised the following question: "What is the extremum value for the lower bounds of this surface area?" There have been obtained some lower bounds (not sharp) for this surface area of the unit ball in $d-$dimensional Minkowski spaces. In [12] (see also [13]), Thompson showed that $\nu_B(\partial B) \geq 2\epsilon_{d-1}$, and $\nu_B(\partial B) \geq (d\epsilon_d)\left(\frac{m_d}{\epsilon_d^2}\right)^{1/d}$, where $m_d := \min\{\lambda(B)\lambda(B^\circ) : B$ a centered symmetric convex body in $\mathbf{R}^d\}$. In [12], Thompson also conjectured that for $d > 3$ the quantity $\nu_B(\partial B)$ is minimal for an ellipsoid.

One goal of this paper is to establish some lower bounds on $\nu_B(\partial B)$ when $d \geq 3$. We will also prove that Thompson's conjecture is valid when the unit ball possesses a certain property.

Furthermore, we shall show that in general Busemann's intersection inequality cannot be strengthened to

$$\lambda^{d-1}(K)\lambda((IK)^\circ) \geq \left(\frac{\epsilon_d}{\epsilon_{d-1}}\right)^d$$

in $\mathbf{R}^d$. Namely, we present a counterexample to this inequality in $\mathbf{R}^3$. This result shows that the "duality" resemblance between projection and intersection inequalities does not always hold (cf. Petty's projection inequality in Section 2).

We shall also show the relationship between the Busemann definition of surface area and cross-section measures.

2. Definitions and Notations

One can find all these notions in the books of Gardner [5] and Thompson [13].

Recall that a *convex body* K is a compact, convex set with nonempty interior, and that K is said to be *centered* if it is centrally symmetric with respect to the origin 0 of $\mathbf{R}^d$. As usual, we denote by S^{d-1} the standard Euclidean unit sphere in $\mathbf{R}^d$. We write λ_i for an *i-dimensional Lebesgue measure* in $\mathbf{R}^d$, where $1 \leq i \leq d$, and instead of λ_d we simply write λ. If $u \in S^{d-1}$, we denote by $u^\perp$ the $(d-1)$-dimensional subspace orthogonal to u, and by l_u the line through the origin parallel to u.

For a convex body K in $\mathbf{R}^d$, we define the *polar body* K° of K by

$$K^\circ = \{y \in \mathbf{R}^d : \langle x, y\rangle \leq 1, x \in K\}.$$

We identify $\mathbf{R}^d$ and its *dual space* $\mathbf{R}^{d*}$ by using the standard basis. In that case, λ_i and λ_i^* coincide in $\mathbf{R}^d$.

If K_1 and K_2 are convex bodies in X, and $\alpha_i \geq 0$, $i = 1, 2$, then the *linear combination* (for $\alpha_1 = \alpha_2 = 1$ the *Minkowski sum*) of these convex bodies is defined by

$$\alpha_1 K_1 + \alpha_2 K_2 := \{x : x = \alpha_1 x_1 + \alpha_2 x_2, x_i \in K_i\}.$$

It is easy to show that the linear combination of convex bodies is itself a convex body.

If K is a convex body in $\mathbf{R}^d$, then the *support function* h_K of K is defined by

$$h_K(u) = \sup\{\langle u, y\rangle : y \in K\}, \quad u \in S^{d-1},$$

giving the distance from 0 to the supporting hyperplane of K with the outward normal u. Note that $K_1 \subset K_2$ if and only if $h_{K_1} \leq h_{K_2}$ for any $u \in S^{d-1}$.

It turns out that every support function is sublinear, and conversely that every sublinear function is the support function of some convex set (see [13, p. 52]).

If $0 \in K$, then the *radial function* of K, $\rho_K(u)$, is defined by

$$\rho_K(u) = \max\{\alpha \geq 0 : \alpha u \in K\},\ u \in S^{d-1},$$

giving the distance from 0 to $l_u \cap \partial K$ in the direction u. Note that $K_1 \subset K_2$ if and only if $\rho_{K_1} \leq \rho_{K_2}$ for any $u \in S^{d-1}$. Both functions have the property that for $\alpha_1, \alpha_2 \geq 0$

$$h_{\alpha_1 K_1 + \alpha_2 K_2}(u) = \alpha_1 h_{K_1}(u) + \alpha_2 h_{K_2}(u)\,,$$
$$\rho_{\alpha_1 K_1 + \alpha_2 K_2}(u) \geq \alpha_1 \rho_{K_1}(u) + \alpha_2 \rho_{K_2}(u)$$

for any direction u.

We mention the relation

$$\rho_{K^\circ}(u) = \frac{1}{h_K(u)}\,, \qquad u \in S^{d-1}\,, \tag{2.1}$$

between the support function of a convex body K and the inverse of the radial function of K°.

For convex bodies K_1, ..., K_{n-1}, K_n in $\mathbf{R}^d$ we denote by $V(K_1, \ldots, K_n)$ their *mixed volume*, defined by

$$V(K_1, ..., K_n) = \frac{1}{d} \int_{S^{d-1}} h_{K_n} dS(K_1, ..., K_{n-1}, u)$$

with $dS(K_1, ..., K_{n-1}, \cdot)$ as the *mixed surface area element* of K_1,...,K_{n-1}.

Note that we have $V(K_1, K_2, ..., K_n) \leq V(L_1, K_2, ..., K_n)$ if $K_1 \subset L_1$, that $V(\alpha K_1, ..., K_n) = \alpha V(K_1, ..., K_n)$, if $\alpha \geq 0$ and that $V(K, K, ..., K) = \lambda(K)$. Furthermore, we will write $V(K[d-1], L)$ instead of $V(\underbrace{K, K, ..., K}_{d-1}, L)$.

Minkowski's inequality for mixed volumes states that if K_1 and K_2 are convex bodies in $\mathbf{R}^d$, then

$$V^d(K_1[d-1], K_2) \geq \lambda^{d-1}(K_1)\lambda(K_2)$$

with equality if and only if K_1 and K_2 are homothetic.

If K_2 is the standard unit ball in $\mathbf{R}^d$, then this inequality becomes the standard isoperimetric inequality.

One of the fundamental theorems on convex bodies refers to the *Blaschke-Santalo inequality* and states that if K is a symmetric convex body in $\mathbf{R}^d$, then

$$\lambda(K)\lambda(K^\circ) \leq \epsilon_d^2$$

with equality if and only if K is an ellipsoid.

The sharp lower bound is known only for zonoids. It is called the *Mahler-Reisner Theorem* which states that if K is a zonoid in $\mathbf{R}^d$, then

$$\frac{4^d}{d!} \leq \lambda(K)\lambda^*(K^\circ)$$

with equality if and only if K is a parallelotope.

Recall that *zonoids* are the limits of zonotopes with respect to the Hausdorff metric, and *zonotopes* are finite Minkowski sums of centered line segments.

For a convex body K in $\mathbf{R}^d$ and $u \in S^{d-1}$ we denote by $\lambda_{d-1}(K|u^\perp)$ the $(d-1)$-dimensional volume of the projection of K onto a hyperplane orthogonal to u. Recall that $\lambda_{d-1}(K|u^\perp)$ is called the $(d-1)$-dimensional *outer cross-section measure* or *brightness* of K at u.

The *projection body* ΠK of a convex body K in $\mathbf{R}^d$ is defined as the body whose support function is given by

$$h_{\Pi K}(u) = \lim_{\varepsilon \to 0} \frac{\lambda(K + \varepsilon[u]) - \lambda(K)}{\varepsilon} = \lambda_{d-1}(K \mid u^\perp),$$

where $[u]$ is the line segment joining $-\frac{u}{2}$ to $\frac{u}{2}$.

Note that $\Pi K = \Pi(-K)$, and that a projection body is a centered zonoid. If K_1 and K_2 are centered convex bodies in $\mathbf{R}^d$ and $\Pi K_1 = \Pi K_2$, then $K_1 = K_2$.

If K is a convex body in $\mathbf{R}^d$, then

$$\binom{2d}{d} d^{-d} \leq \lambda^{d-1}(K)\lambda((\Pi K)^\circ) \leq \left(\frac{\epsilon_d}{\epsilon_{d-1}}\right)^d$$

with equality on the right side if and only if K is an ellipsoid, and with equality on the left side if and only if K is a simplex.

The right side of this inequality is called the *Petty projection inequality*, and the left side was established by Zhang (see [5]).

3. SURFACE AREA AND ISOPERIMETRIX

Let $(\mathbf{R}^d,\ ||\cdot||) = \mathbf{M}^d$ be a d-dimensional real normed linear space, i.e., a *Minkowski space* with *unit ball* B which is a centered convex body. The *unit sphere* of $\mathbf{M}^d$ is the boundary of the unit ball and denoted by ∂B.

A Minkowski space $\mathbf{M}^d$ possesses a Haar measure ν (or ν_B if we need to emphasize the norm), and this measure is unique up to multiplication of the Lebesgue measure by a constant, i.e.,

$$\nu = \sigma_B \lambda.$$

It turns out that it is not as easy a problem to choose a right multiple σ as it seems. These two measures ν and λ have to coincide in the standard Euclidean space.

Definition 3.1. *If K is a convex body in R^d, then the d-*dimensional Busemann volume *of K is defined by*

$$\nu_B(K) = \frac{\epsilon_d}{\lambda(B)}\lambda(K), \text{ i.e.}, \quad \sigma_B = \frac{\epsilon_d}{\lambda(B)}.$$

Note that these definitions coincide with the standard notion of volume if the space is Euclidean, and that $\nu_B(B) = \epsilon_d$.

Let M be a surface in $\mathbf{R}^d$ with the property that at each point x of M there is a unique tangent hyperplane, and that u_x is the unit normal vector to this hyperplane at x. Then the *Minkowski surface area* of M is defined by

$$\nu_B(M) := \int_M \sigma_B(u_x) dS(x).$$

For the *Busemann surface area,* $\sigma_B(u)$ is defined by

$$\sigma_B(u) = \frac{\epsilon_{d-1}}{\lambda(B \cap u^\perp)}.$$

The function $\sigma(u)$ can be extended homogeneously to the whole of $\mathbf{M}^d$, and it turns out that this extended function is convex (see [4], or [5]). Thus, this extended function σ is the support function of some convex body in $\mathbf{R}^d$. We denote this convex body by T_B, therefore if K is a convex body in $\mathbf{M}^d$, then Minkowski's surface area of K can also be defined by

$$\nu_B(\partial K) = dV(K[d-1],\ T_B). \tag{3.1}$$

We deduce that $\nu_B(\partial T_B) = d\lambda(T_B)$.

From Minkowski's inequality for mixed volumes one can see that T_B plays a central role regarding the solution of the *isoperimetric problem* in Minkowski spaces.

Among the homothetic images of T_B we want to specify a unique one, called the *isoperimetrix* $\hat{T}_B$, determined by $\nu_B(\partial \hat{T}_B) = d\nu_B(\hat{T}_B)$.

Proposition 3.1. *If B is the unit ball of $\mathbf{M}^d$ and $\hat{T}_B = \frac{\lambda(B)}{\epsilon_d} T_B$, then*

$$\nu_B(\partial \hat{T}_B) = d\nu_B(\hat{T}_B).$$

Proof. We use properties of the surface area, and straight calculation to obtain

$$\begin{aligned}\nu_B(\partial\hat{T}_B) &= \frac{\lambda^{d-1}(B)}{\epsilon_d^{d-1}}\nu_B(\partial T_B)\\ &= d\frac{\lambda^{d-1}(B)}{\epsilon_d^{d-1}}\lambda(T_B)\\ &= d\frac{\epsilon_d}{\lambda(B)}\lambda(\hat{T}_B) = d\nu_B(\hat{T}_B).\end{aligned}$$

□

Now we define the inner and outer radius of a convex body in a Minkowski space. Note that in Minkowski geometry these two notions are used with different meanings (see [11], [13]). As in [13], here these notions are defined by using the isoperimetrix.

Definition 3.2. *If* K *is a convex body in* $\mathbf{R}^d$*, the* inner radius *of* K*,* $r(K)$*, is defined by*

$$r(K) = \max\{\alpha : \exists x \in \mathbf{M}^d \textit{with} \alpha\hat{T}_B \subseteq K + x\},$$

and the outer radius *of* K*,* $R(K)$*, is defined by*

$$R(K) = \min\{\alpha : \exists x \in \mathbf{M}^d \textit{with} \alpha\hat{T}_B \supseteq K + x\}.$$

4. THE INTERSECTION BODY

We know that $\sigma_B(f) = \frac{\epsilon_{d-1}}{\lambda(B\cap u^\perp)}$ is a convex function and the support function of T_B. Since the support function is the inverse of the radial function, we have that

$$\rho(u) = \sigma_B^{-1}(u) = \epsilon_{d-1}^{-1}\lambda(B \cap u^\perp)$$

is the radial function of T_B°.

The *intersection body* of K is a convex body whose radial function is $\lambda(K \cap u^\perp)$ in a given direction u, and we denote it by IK (see [7] for more about intersection bodies). We can also rewrite the solution of the isoperimetric problem T_B as

$$T_B = \epsilon_{d-1}(IB)^\circ. \tag{4.1}$$

One can see that $T_{\alpha B} = \alpha^{1-d}T_B$ for $\alpha \geq 0$.

There is an important relationship between the volume of a convex body and the volume of its intersection body. It is called *Busemann's intersection inequality* which states that if K is a convex body in $\mathbf{R}^d$, then

$$\lambda(IK) \leq \left(\frac{\epsilon_{d-1}}{\epsilon_d}\right)^d \epsilon_d^2\lambda^{d-1}(K)$$

with equality if and only if K is a centered ellipsoid (see [5]).

Setting $K = B$ in Busemann's intersection inequality and using (4.1), we can rewrite this inequality as

$$\lambda(T_B^\circ)\epsilon_d^d \leq \epsilon_d^2\lambda^{d-1}(B). \tag{4.2}$$

It turns out that if K is a convex body in X with 0 as an interior point, then

$$IK \subseteq \Pi K, \tag{4.3}$$

with equality if and only if K is a centered ellipsoid (see [8]).

Recall that the intersection body of a centered d-dimensional ellipsoid E is a centered ellipsoid, i.e., more precisely we have

$$IE = \frac{\epsilon_{d-1}\lambda(E)}{\epsilon_d}E.$$

5. Some Lower Bounds on the Surface Area of the Unit Ball

As we mentioned in the introduction, the reasonable question is to ask how large and how small the surface area of the unit ball of $\mathbf{M}^d$ for the Busemann definition can be. In [4] Busemann and Petty showed that if B is the unit ball of a d-dimentional Minkowski space $\mathbf{M}^d$, then

$$\nu_B(\partial B) \leq 2d\epsilon_{d-1}$$

with equality if and only if B is a parallelotope.

In this section we establish lower bounds for the Busemann surface area of the unit ball in a d-dimensional Minkowski space when $d \geq 3$.

Theorem 5.1. *If B is the unit ball of a $d-$dimensional Minkowski space $\mathbf{M}^d$, then*

$$\nu_B(\partial B) \geq \epsilon_{d-1}\binom{2d}{d}^{\frac{1}{d}}.$$

Proof. Since $T_B = \epsilon_{d-1}(IB)^\circ \supseteq \epsilon_{d-1}(\Pi B)^\circ$, we get by Zhang's inequality

$$\lambda(T_B) \geq \epsilon_{d-1}^d \lambda((\Pi B)^\circ) \geq \binom{2d}{d} d^{-d}\epsilon_{d-1}^d \lambda^{1-d}(B).$$

Therefore

$$d^d\lambda^{d-1}(B)\lambda(T_B) \geq \binom{2d}{d}\epsilon_{d-1}^d.$$

From Minkowski's inequality it follows that $\nu_B^d(\partial B) \geq d^d\lambda^{d-1}(B)\lambda(T_B)$. Hence the result follows. □

We note that $\binom{2d}{d} \geq 2^d$.

Theorem 5.2. *If B is the unit ball of a d-dimensional Minkowski space $\mathbf{M}^d$, then*

$$\nu_B(\partial B) \geq d\epsilon_d\left(\frac{\lambda(T_B)\lambda(T_B^\circ)}{\epsilon_d^2}\right)^{\frac{1}{d}}$$

with equality if and only if B is an ellipsoid.

Proof. It follows from Busemann's intersection inequality that

$$\lambda(T_B^\circ) \leq (\epsilon_d^2/\epsilon_d^d)\lambda^{d-1}(B).$$

Therefore

$$\lambda(T_B)\lambda(T_B^\circ) \leq (\epsilon_d^2/\epsilon_d^d)\lambda^{d-1}(B)\lambda(T_B).$$

Using Minkowski's inequality we get

$$\frac{\nu_B^d(\partial B)}{d^d\epsilon_d^d}\epsilon_d^2 \geq \lambda(T_B)\lambda(T_B^\circ).$$

Hence the inequality follows, and one can also see that equality holds if and only if B is an ellipsoid. □

Let us define $\mu_{T_B}(T_B) = \frac{\lambda(T_B)\lambda(T_B^\circ)}{\epsilon_d}$, i.e., the Holmes-Thompson definition of volume for T_B (see [6] or [13]) in a d-dimensional Minkowski space $(\mathbf{R}^d,\ T_B)$.

It follows from the Blaschke-Santalo inequality that

$$\epsilon_d \left(\frac{\lambda(T_B)\lambda(T_B^\circ)}{\epsilon_d^2} \right)^{\frac{1}{d}} \geq \mu_{T_B}(T_B)$$

with equality if and only if B is an ellipsoid.

We obtain the following.

Corollary 5.3. *If B is the unit ball of a d-dimensional Minkowski space M^d, then*

$$\nu_B(\partial B) \geq d\mu_{T_B}(T_B),$$

with equality if and only if B is an ellipsoid.

We show that Thompson's conjecture is valid when the unit ball possesses a certain property.

Theorem 5.4. *If B is the unit ball of M^d with an outer radius of $R(B)$, then*

$$\nu_B(\partial B) \geq \frac{d\epsilon_d}{R},$$

with equality if and only if $B = R(B)\hat{T}_B$.

Proof. Since $\hat{T}_B$ is the solution of the isoperimetric problem, we have

$$\frac{\nu_B^d(\partial B)}{\nu_B^{d-1}(B)} \geq \frac{\nu_B^d(\partial \hat{T}_B)}{\nu_B^{d-1}(\hat{T}_B)} = d^d \nu_B(\hat{T}_B) \geq \frac{d^d}{R^d}\nu_B(B).$$

Hence the result follows, since $\nu_B(B) = \epsilon_d$. Obviously, if equality holds, then we get $B = R(B)\hat{T}_B$. If $B = R(B)\hat{T}_B$, then we have

$$\nu_B(\partial B) = R^{d-1}\nu_B(\partial \hat{T}_B) = \frac{d}{R}R^d \nu_B(\hat{T}_B) = \frac{d}{R}\nu_B(B).$$

□

Corollary 5.5. *If B is the unit ball of a $d-$dimensional Minkowski space M^d such that $R(B) \leq 1$, then*

$$\nu_B(\partial B) \geq d\epsilon_d,$$

with equality if and only if $B = \hat{T}_B$.

Proof. The inequality part and the implication follow from Theorem 5.4.

Now assume that $R(B) \leq 1$ and $\nu_B(\partial B) = d\epsilon_d$. Then $B \subseteq \hat{T}_B$, and

$$d^d \epsilon_d^d = d^d V^d(B[d-1], T_B) \geq d^d \lambda^{d-1}(B)\lambda(T_B).$$

This gives us that $\lambda(B) \geq \lambda(\hat{T}_B)$. Hence $\lambda(B) = \lambda(\hat{T}_B)$, and this is the case when $B = \hat{T}_B$. □

In [12], Thompson showed that if the unit ball is an affine regular rombic dodecahedron in $\mathbf{R}^3$, then $\nu_B(\partial B) = d\epsilon_d = 4\pi$. Therefore, for a rombic dodecahedron in $\mathbf{R}^3$ either $B = \hat{T}_B$ or $R(B) > 1$. The first one cannot be the case, since if B is a rombic dodecahedron, then the facets of $(IB)^\circ$ become "round" (cf. [13, p. 153]).

Corollary 5.6. *If $R(B)$ is the outer radius of the unit ball of B in a d-dimensional Minkowski space M^d, then*

$$R(B) \geq \frac{\epsilon_d}{2\epsilon_{d-1}}.$$

Proof. The result follows from the fact that $\nu_B(\partial B) \leq 2d\epsilon_{d-1}$ and Theorem 5.4. □

In [9], it was proved that $R(B) \leq \frac{d\epsilon_d}{2\epsilon_{d-1}}$ with equality if and only if B is a parallelotope.

Theorem 5.7. *If B is the unit ball of a d-dimensional Minkowski space $\mathbf{M}^d$ such that $\lambda(\hat{T}_B) \geq \lambda(B)$, then*

$$\nu_B(\partial B) \geq d\epsilon_d,$$

with equality if and only if $B = \hat{T}_B$.

Proof. We can rewrite $\lambda(\hat{T}_B) \geq \lambda(B)$ as

$$\lambda^{d-1}(B)\lambda(T_B) \geq \epsilon_d^d.$$

This gives us

$$\nu_B^d(\partial B) = d^d V^d(B[d-1], T_B) \geq d^d \lambda^{d-1}(B)\lambda(T_B) \geq d^d \epsilon_d^d.$$

Hence the result follows. Obviously, if $B = \hat{T}_B$, then $\nu_B(\partial B) = d\epsilon_d$. If $\nu_B(\partial B) = d\epsilon_d$, then it follows from Minkowski's inequality that B and T_B must be homothetic. Therefore $\lambda(\hat{T}_B) = \lambda(B)$, and this is the case when $B = \hat{T}_B$. □

From Theorem 5.7 it follows that if B is a rombic dodecahedron in $\mathbf{M}^d$, then $\lambda(\hat{T}_B) < \lambda(B)$.

In [10] it was conjectured that if $\hat{I}_B$ is the isoperimetrix for the Holmes-Thompson definition in a d-dimensional Minkowski space $\mathbf{M}^d$, then

$$\lambda(\hat{I}_B) \geq \lambda(B)$$

with equality if and only if B is an ellipsoid.

Therefore, if B is a rombic dodecahedron in $\mathbf{R}^3$, then $\lambda(\hat{I}_B) > \lambda(\hat{T}_B)$.

Problem 5.1. If $r(B)$ is the inner radius of the unit ball B for the isoperimetrix $\hat{T}_B$, is it then true that

$$r(B) \leq 1$$

with equality if and only if B is an ellipsoid?

The answer of this question will tell us whether there exists a unit ball such that $\hat{T}_B \subseteq B$. For the Holmes-Thompson definition of the isoperimetrix $\hat{I}_B$, $r(B) \leq 1$ holds with equality if and only if B is an ellipsoid (see [10] or [13]).

In [13] (Problem 7.4.3, or p. 245) A.C. Thompson asked whether Busemann's intersection inequality can be strengthened to

$$\lambda^{d-1}(K)\lambda((IK)^\circ) \geq \left(\frac{\epsilon_d}{\epsilon_{d-1}}\right)^d.$$

It is easy to show that equality holds for an ellipsoid. Setting $K = B$, we get

$$\lambda(\hat{T}_B) \geq \lambda(B).$$

As we have shown, the last inequality does not hold when B is a rombic dodecahedron in $\mathbf{M}^3$.

Now we show the relationship between cross-section measures and the Busemann definition of surface area.

Proposition 5.8. *If the unit ball B of $\mathbf{M}^d$ satisfies*

$$\frac{\lambda_{d-1}(B \cap u^{\perp})\lambda_1(B|l_u)}{\lambda(B)} \leq \frac{2\epsilon_{d-1}}{\epsilon_d}$$

for each $u \in S^{d-1}$, then

$$\nu_B(\partial B) \geq d\epsilon_d.$$

Proof. It follows from the hypothesis of the proposition that for any $u \in S^{d-1}$

$$\rho_{IB}(u)h_B(u) \leq \frac{\epsilon_{d-1}}{\epsilon_d}\lambda(B).$$

Using (2.1), we get $\epsilon_d h_B(u) \leq \lambda(B)h_{T_B}(u)$ for each direction, and therefore

$$\epsilon_d B \subseteq T_B \lambda(B).$$

Hence the result follows from properties of mixed volumes and (3.1). □

Problem 5.2.

a) Does there exist a centered convex body K in $\mathbf{R}^d$ such that

$$\frac{\lambda_{d-1}(K \cap u^{\perp})\lambda_1(K|l_u)}{\lambda(K)} > \frac{2\epsilon_{d-1}}{\epsilon_d}$$

for each $u \in S^{d-1}$?

b) Is it true that for a centered convex body K in $\mathbf{R}^d$

$$\frac{\lambda_{d-1}(K \cap u^{\perp})\lambda_1(K|l_u)}{\lambda(K)} = \frac{2\epsilon_{d-1}}{\epsilon_d}$$

holds for each $u \in S^{d-1}$ only when K is an ellipsoid?

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