SUBORDINATION AND SUPERORDINATION RESULTS FOR Φ-LIKE FUNCTIONS

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Abstract: Let q_1 be convex univalent and q_2 be univalent in $\Delta := \{z : |z| < 1\}$ with $q_1(0) = q_2(0) = 1$. Let f be a normalized analytic function in the open unit disk Δ. Let Φ be an analytic function in a domain containing $f(\Delta)$, $\Phi(0) = 0$ and $\Phi'(0) = 1$. We give some applications of first order differential subordination and superordination to obtain sufficient conditions for the function f to satisfy

$$q_1(z) \prec \frac{z(f*g)'(z)}{\Phi(f*g)(z)} \prec q_2(z)$$

where g is a fixed function.

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1. Introduction and Motivations

Let $\mathcal{A}$ be the class of all normalized analytic functions $f(z)$ in the open unit disk $\Delta := \{z : |z| < 1\}$ satisfying $f(0) = 0$ and $f'(0) = 1$. Let $\mathcal{H}$ be the class of functions analytic in Δ and for any $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots$. Let $p, h \in \mathcal{H}$ and let $\phi(r,s,t;z) : \mathbb{C}^3 \times \Delta \to \mathbb{C}$. If p and $\phi(p(z), zp'^2 p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \phi(p(z), zp'^2 p''(z); z), \tag{1.1}$$

then p is a solution of the differential superordination (1.1). If f is subordinate to F, then F is called a superordinate of f. An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (1.1). A univalent subordinant $\bar{q}$ that satisfies $q \prec \bar{q}$ for all subordinants q of (1.1) is said to be the best subordinant. Recently Miller and Mocanu [5] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'^2 p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.2}$$

Using the results of Miller and Mocanu [4], Bulboacă [2] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [1]. In an earlier investigation, Shanmugam et al. [8] obtained sufficient conditions for a normalized analytic function $f(z)$ to satisfy $q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$ and $q_1(z) \prec \frac{z^2 f'(z)}{\{f(z)\}^2} \prec q_2(z)$ where q_1 and q_2 are given univalent functions in Δ with $q_1(0) = 1$ and $q_2(0) = 1$. A systematic study of the subordination and superordination has been studied very recently by Shanmugam *et al.* in [9] and [10] (see also the references cited by them).

Let Φ be an analytic function in a domain containing $f(\Delta)$ with $\Phi(0) = 0$ and $\Phi'(0) = 1$. For any two analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) =$

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$\sum_{n=0}^{\infty} b_n z^n$, the Hadamard product or convolution of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

The function $f \in \mathcal{A}$ is called Φ-like if

$$\Re\left(\frac{zf'(z)}{\Phi(f(z))}\right) > 0 \quad (z \in \Delta). \tag{1.3}$$

The concept of $\Phi-$ like functions was introduced by Brickman [3] and he established that a function $f \in \mathcal{A}$ is univalent if and only if f is Φ-like for some Φ. For $\Phi(w) = w$, the function f is starlike. In a later investigation, Ruscheweyh [7] introduced and studied the following more general class of Φ-like functions.

Definition 1.1. Let Φ be analytic in a domain containing $f(\Delta)$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(w) \neq 0$ for $w \in f(\Delta) \setminus \{0\}$. Let $q(z)$ be a fixed analytic function in Δ, $q(0) = 1$. The function $f \in \mathcal{A}$ is called Φ-like with respect to q if

$$\frac{zf'(z)}{\Phi(f(z))} \prec q(z) \quad (z \in \Delta). \tag{1.4}$$

When $\Phi(w) = w$, we denote the class of all Φ-like functions with respect to q by $S^*(q)$.

Using the definition of $\Phi-$ like functions, we introduce the following class of functions.

Definition 1.2. Let g be a fixed function in $\mathcal{A}$. Let Φ be analytic in a domain containing $f(\Delta)$, $\Phi(0) = 0$, $\Phi'(0) = 1$ and $\Phi(w) \neq 0$ for $w \in f(\Delta) \setminus \{0\}$. Let $q(z)$ be a fixed analytic function in Δ, $q(0) = 1$. The function $f \in \mathcal{A}$ is called Φ-like with

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respect to $S_g^*(q)$ if

$$\frac{z(f*g)'(z)}{\Phi(f*g)(z)} \prec q(z) \quad (z \in \Delta). \tag{1.5}$$

We note that $S^*_{\frac{z}{1-z}}(q) := S^*(q)$.

In the present investigation, we obtain sufficient conditions for a normalized analytic function f to satisfy

$$q_1(z) \prec \frac{z(f*g)'(z)}{\Phi(f*g)(z)} \prec q_2(z).$$

We shall need the following definition and results to prove our main results. In this sequel, unless otherwise stated, α and γ are complex numbers.

Definition 1.3 *([4, Definition 2, p. 817]).* Let Q be the set of all functions f that are analytic and injective on $\bar{\Delta} - E(f)$, where

$$E(f) = \left\{ \zeta \in \partial\Delta : \lim_{z\to\zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial\Delta - E(f)$.

Lemma 1.1 ([4, Theorem 3.4h, p. 132]). *Let q be univalent in the open unit disk Δ and θ and ϕ be analytic in a domain D containing $q(\Delta)$ with $\phi(\omega) \neq 0$ when $\omega \in q(\Delta)$. Set $\xi(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + \xi(z)$. Suppose that*

1. *$\xi(z)$ is starlike univalent in Δ, and*
2. $\Re\left\{\frac{zh'(z)}{\xi(z)}\right\} > 0$ $(z \in \Delta)$.

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If p is analytic in Δ with $p(\Delta) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{1.6}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 1.2. *[2, Corollary 3.1, p. 288] Let q be univalent in Δ, ϑ and φ be analytic in a domain D containing $q(\Delta)$. Suppose that*

1. $\Re\left[\frac{\vartheta'(q(z))}{\varphi(q(z))}\right] > 0$ *for $z \in \Delta$, and*
2. $\xi(z) = zq'(z)\varphi(q(z))$ *is starlike univalent function in Δ.*

If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\Delta) \subset D$, and $\vartheta(p(z)) + zp'(z)\varphi(p(z))$ is univalent in Δ, and

$$\vartheta(q(z)) + zq'(z)\varphi(q(z)) \prec \vartheta(p(z)) + zp'(z)\varphi(p(z)), \tag{1.7}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

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2. Main Results

By making use of Lemma 1.1, we prove the following result.

Theorem 2.1. *Let* $q(z) \neq 0$ *be analytic and univalent in* Δ *with* $q(0) = 1$ *such that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* Δ*. Let* $q(z)$ *satisfy*

$$\Re\left[1 + \frac{\alpha q(z)}{\gamma} - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right] > 0. \tag{2.1}$$

Let

$$\Psi(\alpha, \gamma, g; z) := \alpha\left\{\frac{z(f*g)'(z)}{\Phi(f*g)(z)}\right\} + \gamma\left\{1 + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{z\left(\Phi(f*g)(z)\right)'}{\Phi(f*g)(z)}\right\}. \tag{2.2}$$

If q *satisfies*

$$\Psi(\alpha, \gamma, g; z) \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)}, \tag{2.3}$$

then

$$\frac{z(f*g)'(z)}{\Phi(f*g)(z)} \prec q(z)$$

and q *is the best dominant.*

Proof. Let the function $p(z)$ be defined by

$$p(z) := \frac{z(f*g)'(z)}{\Phi(f*g)(z)}. \tag{2.4}$$

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Then the function $p(z)$ is analytic in Δ with $p(0) = 1$. By a straightforward computation

$$\frac{zp'(z)}{p(z)} = \left\{1 + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{z[\Phi(f*g)(z)]'}{\Phi(f*g)(z)}\right\}$$

which, in light of hypothesis (2.3) of Theorem 2.1, yields the following subordination

$$\alpha p(z) + \frac{\gamma z p'(z)}{p(z)} \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)}. \tag{2.5}$$

By setting

$$\theta(\omega) := \alpha\omega \quad \text{and} \quad \phi(\omega) := \frac{\gamma}{\omega},$$

it can be easily observed that $\theta(\omega)$ and $\phi(\omega)$ are analytic in $\mathbb{C} \setminus \{0\}$ and that

$$\phi(\omega) \neq 0 \quad (\omega \in \mathbb{C} \setminus \{0\}).$$

Also, by letting

$$\xi(z) = zq'(z)\phi(q(z)) = \frac{\gamma}{q(z)} zq'(z). \tag{2.6}$$

and

$$h(z) = \theta\{q(z)\} + \xi(z) = \alpha q(z) + \frac{\gamma}{q(z)} zq'(z), \tag{2.7}$$

we find that $\xi(z)$ is starlike univalent in Δ and that

$$\Re\left[1 + \frac{\alpha q(z)}{\gamma} - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right] > 0$$

by the hypothesis (2.1). The assertion of Theorem 2.1 now follows by an application of Lemma 1.1. ∎

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When $\Phi(\omega) = \omega$ in Theorem 2.1 we get:

Corollary 2.2. *Let $q(z) \neq 0$ be univalent in Δ with $q(0) = 1$. If q satisfies*

$$(\alpha - \gamma)\frac{z(f*g)'(z)}{(f*g)(z)} + \gamma\left\{1 + \frac{z(f*g)''(z)}{(f*g)'(z)}\right\} \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)},$$

then

$$\frac{z(f*g)'(z)}{(f*g)(z)} \prec q(z)$$

and q is the best dominant.

For $g(z) = \frac{z}{1-z}$ and $\Phi(\omega) = \omega$, we get the following corollary.

Corollary 2.3. *Let $q(z) \neq 0$ be univalent in Δ with $q(0) = 1$. If q satisfies*

$$(\alpha - \gamma)\frac{zf'(z)}{f(z)} + \gamma\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)},$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z)$$

and q is the best dominant.

For the choice $\alpha = \gamma = 1$ and $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$ in Corollary 2.3, we have the following result of Ravichandran and Jayamala [6].

Corollary 2.4. *If $f \in \mathcal{A}$ and*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Az)(1+Bz)},$$

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then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ *is the best dominant.*

Theorem 2.5. *Let* $\gamma \neq 0$*. Let* $q(z) \neq 0$ *be convex univalent in* Δ *with* $q(0) = 1$ *such that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* Δ*. Suppose that* $q(z)$ *satisfies*

$$\Re\left[\frac{\alpha q(z)}{\gamma}\right] > 0. \tag{2.8}$$

If $f \in \mathcal{A}$, $\frac{z(f*g)'(z)}{\Phi(f*g)(z)} \in \mathcal{H}[1,1] \cap Q$, $\Psi(\alpha,\gamma,g;z)$ *as defined by (2.2) is univalent in* Δ *and*

$$\alpha q(z) + \frac{\gamma z q'(z)}{q(z)} \prec \Psi(\alpha,\gamma,g;z), \tag{2.9}$$

then

$$q(z) \prec \frac{z(f*g)'(z)}{\Phi(f*g)(z)}$$

and q *is the best subordinant.*

Proof. By setting

$$\vartheta(w) := \alpha w \quad \text{and} \quad \varphi(w) := \frac{\gamma}{\omega},$$

it is easily observed that $\vartheta(w)$ is analytic in $\mathbb{C}$, $\varphi(w)$ is analytic in $\mathbb{C}\setminus\{0\}$ and that

$$\varphi(w) \neq 0, \quad (w \in \mathbb{C}\setminus\{0\}).$$

The assertion of Theorem 2.5 follows by an application of Lemma 1.2. ∎

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For $\Phi(\omega) = \omega$ in Theorem 2.5, we get

Corollary 2.6. *Let $q(z) \neq 0$ be convex univalent in Δ with $q(0) = 1$. If $f \in \mathcal{A}$ and*

$$\alpha q(z) + \frac{\gamma z q'(z)}{q(z)} \prec (\alpha - \gamma)\left\{\frac{z(f*g)'(z)}{(f*g)(z)}\right\} + \gamma\left\{1 + \frac{z(f*g)''(z)}{(f*g)'(z)}\right\},$$

then

$$q(z) \prec \frac{z(f*g)'(z)}{(f*g)(z)}$$

and q is the best subordinant.

Combining Theorem 2.1 and Theorem 2.5 we get the following sandwich theorem.

Theorem 2.7. *Let q_1 be convex univalent and q_2 be univalent in Δ satisfying (2.8) and (2.1) respectively such that $q_1(0) = 1$, $q_2(0) = 1$, $\frac{zq_1'(z)}{q_1(z)}$ and $\frac{zq_2'(z)}{q_2(z)}$ are starlike univalent in Δ with*

$$q_1(z) \neq 0 \quad \text{and} \quad q_2(z) \neq 0.$$

Let $f \in \mathcal{A}$, $\frac{z(f*g)'(z)}{\Phi(f*g)(z)} \in \mathcal{H}[1,1] \cap Q$, and $\Psi(\alpha, \gamma, g; z)$ as defined by (2.2) be univalent in Δ. Further, if

$$\alpha q_1(z) + \frac{\gamma z q_1'(z)}{q_1(z)} \prec \Psi(\alpha, \gamma, g; z) \prec \alpha q_2(z) + \frac{\gamma z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{z(f*g)'(z)}{\Phi(f*g)(z)} \prec q_2(z)$$

and q_1 and q_2 are respectively the best subordinant and best dominant.

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