journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 9 (2008), Issue 4, Article 104, 5 pp.

A MULTINOMIAL EXTENSION OF AN INEQUALITY OF HABER

HACÈNE BELBACHIR

USTHB/ FACULTÉ DE MATHÉMATIQUES
BP 32, EL ALIA, 16111 BAB EZZOUAR
ALGER, ALGERIA.
hbelbachir@usthb.dz

Received 10 July, 2008; accepted 17 September, 2008
Communicated by L. Tóth

ABSTRACT. In this paper, we establish the following: Let $a_1, a_2, \ldots, a_m$ be non negative real numbers, then for all $n \geq 0$, we have

$$\frac{1}{\binom{n+m-1}{m-1}} \sum_{i_1+i_2+\cdots+i_m=n} a_1^{i_1} a_2^{i_2} \cdots a_m^{i_m} \geq \left(\frac{a_1+a_2+\cdots+a_m}{m}\right)^n.$$

The case $m = 2$ gives the Haber inequality. We apply the result to find lower bounds for the sum of reciprocals of multinomial coefficients and for symmetric functions.

Key words and phrases: Haber inequality, multinomial coefficient, symmetric functions.

2000 *Mathematics Subject Classification.* 05A20, 05E05.

1. INTRODUCTION

In 1978, S. Haber [3] proved the following inequality: Let a and b be non negative real numbers, then for every $n \geq 0$, we have

$$\frac{1}{n+1}\left(a^n + a^{n-1}b + \cdots + ab^{n-1} + b^n\right) \geq \left(\frac{a+b}{2}\right)^n. \tag{1.1}$$

Another formulation of (1.1) is

$$f(x,y) \geq f\left(\tfrac{1}{2}, \tfrac{1}{2}\right) \text{ for all non negative numbers } x, y \text{ satisfying } x+y=1,$$

where

$$f(x,y) = \sum_{i+j=n} x^i y^j \quad \text{with } x = \frac{a}{a+b} \text{ and } y = \frac{b}{a+b}.$$

In 1983 [5], A. Mc.D. Mercer, using an analogous technique, gave an extension of Haber's inequality for convex sequences.

Special thanks to A. Chabour, S. Y. Raffed, and R. Souam for useful discussions. The proof given in the remark is due to A. Chabour.
201-08

Let $(u_k)_{0\leq k\leq n}$ be a convex sequence, then the following inequality holds

$$\frac{1}{n+1}\sum_{k=0}^{n}u_k \geq \sum_{k=0}^{n}\binom{n}{k}u_k. \tag{1.2}$$

In 1994 [1], using also the same tools, H. Alzer and J. Pečarić obtained a more general result than the relation (1.2).

In 2004 [6], A. Mc.D. Mercer extended the result using an equivalent inequality of (1.1) as a polynomial in $x=\frac{a}{b}$, and deduced relations satisfying (1.2), see [1].

Let $P(x)=\sum_{k=0}^{n}a_k x^k$ satisfy $P(x)=(x-1)^2 Q(x)$, where the coefficients of $Q(x)$ are real and non negative. Then if $(u_k)_{0\leq k\leq n}$ is a convex sequence, we have

$$\sum_{k=0}^{n}a_k u_k \geq 0. \tag{1.3}$$

Our proposal is to establish an extension of the relation (1.1) to n real numbers.

2. MAIN RESULT

In this section, we give an extension of the inequality given by the relation (1.1) for several variables.

Theorem 2.1 (Generalized Haber inequality)**.** *Let $a_1,a_2,\dots,a_m$ be non negative real numbers, then for all $n\geq 0$, one has*

$$\frac{1}{\binom{n+m-1}{m-1}}\sum_{i_1+i_2+\cdots+i_m=n}a_1^{i_1}a_2^{i_2}\cdots a_m^{i_m} \geq \left(\frac{a_1+a_2+\cdots+a_m}{m}\right)^n. \tag{2.1}$$

For another formulation of (2.1), let us consider the following homogeneous polynomial of degree n

$$f_m(x_1,x_2,\dots,x_m)=\sum_{i_1+i_2+\cdots+i_m=n}x_1^{i_1}x_2^{i_2}\cdots x_m^{i_m}$$

where $x_1,x_2,\dots,x_m$ are non negative real numbers satisfying the constraint $x_1+x_2+\cdots+x_m=1$. By setting for all $i=1,\dots,m$; $x_i=\frac{a_i}{a_1+a_2+\cdots+a_m}$, the inequality given by (2.1) becomes

$$f_m(x_1,x_2,\dots,x_m)\geq f_m\left(\tfrac{1}{m},\tfrac{1}{m},\dots,\tfrac{1}{m}\right) \tag{2.2}$$

Proof. Let $(y_1,y_2,\dots,y_m)$ be the values for which f_m is minimal. It is well known that the gradient of f_m at $(y_1,y_2,\dots,y_m)$ is parallel to that of the constraint which is $(1,1,\dots,1)$, one then deduces

$$\left.\frac{\partial f_m}{\partial x_\alpha}(x_1,\dots,x_m)\right|_{x_\alpha=y_\alpha}=\left.\frac{\partial f_m}{\partial x_\beta}(x_1,\dots,x_m)\right|_{x_\beta=y_\beta},$$

for all α,β, $1\leq\alpha\neq\beta\leq m$, which is equivalent to

$$\sum_{i_1+\cdots+i_m=n}i_\alpha\left(\prod_{\substack{j=1\\ j\neq\alpha,\beta}}^{m}x_j^{i_j}\right)y_\alpha^{i_\alpha-1}y_\beta^{i_\beta}=\sum_{i_1+\cdots+i_m=n}i_\beta\left(\prod_{\substack{j=1\\ j\neq\alpha,\beta}}^{m}x_j^{i_j}\right)y_\alpha^{i_\alpha}y_\beta^{i_\beta-1},$$

i.e.

$$\sum_{i_1+\cdots+i_m=n}\left(\prod_{\substack{j=1\\ j\neq\alpha,\beta}}^{m}x_j^{i_j}\right)y_\alpha^{i_\alpha-1}y_\beta^{i_\beta-1}(i_\alpha y_\beta-i_\beta y_\alpha)=0,$$

which one can write as

$$\sum_{r=0}^{n}\left[\sum_{i_\alpha+i_\beta=r} y_\alpha^{i_\alpha-1}y_\beta^{i_\beta-1}\left(i_\alpha y_\beta - i_\beta y_\alpha\right)\right]\left[\sum_{\substack{i_1+\cdots+i_m=n-r\\ i_k\neq i_\alpha \,\&\, i_k\neq i_\beta}}\left(\prod_{\substack{j=1\\ j\neq\alpha,\beta}}^{m} x_j^{i_j}\right)\right]=0.$$

This last expression is a polynomial of several variables $x_1,\ldots,x_j,\ldots,x_m$ $(j\neq\alpha,\beta)$ which is null if all coefficients are zero. Then for $y_\alpha=a$ and $y_\beta=b$, one obtains for every $r=0,\ldots,n$

$$\sum_{i+j=r} a^{i-1}b^{j-1}\left(ib-ja\right)=0.$$

By developing the sum and gathering the terms of the same power, one obtains

$$\sum_{i=0}^{r-1}\left(2i+1-r\right)a^i b^{r-1-i}=0.$$

By gathering successively the extreme terms of the sum, we have

$$\sum_{i=0}^{\left\lfloor\frac{r+1}{2}\right\rfloor}\left(r-2i-1\right)\left(a^{r-2i-1}-b^{r-2i-1}\right)a^{2i}b^{2i}=0$$

which is equivalent to

$$\left(a-b\right)\sum_{i=0}^{\left\lfloor\frac{r+1}{2}\right\rfloor}\sum_{k=0}^{r-2i-2}\left(r-2i-1\right)a^{k+2i}b^{r-k-2}=0.$$

The double summation is positive, then one deduces that

$$a=b\Longleftrightarrow y_\alpha=y_\beta.$$

The symmetric group $\mathcal{S}_m$ acts naturally by permutations over $\mathbb{R}\left[x_1,x_2,\ldots,x_m\right]$ and leaves invariant $f_m\left(x_1,x_2,\ldots,x_m\right)$ and $x_1+x_2+\cdots+x_m=1$. Finally, one concludes that

$$y_1=y_2=\cdots=y_m=\frac{1}{m}.$$

□

Remark 1. We can prove the above inequality using:

(1) induction over m exploiting Haber's inequality and the well known relation

$$\binom{n}{i_1,i_2,\ldots,i_m}=\binom{n-i_m}{i_1,i_2,\ldots,i_{m-1}}\binom{n}{i_m}.$$

(2) the sectional method for the function

$$f_m\left(x_1,x_2,\ldots,x_m\right)=\sum_{|i|=n}x_1^{i_1}x_2^{i_2}\cdots x_m^{i_m}$$

with the constraint $x_1+x_2+\cdots+x_m=1$;

Let $a_1,a_2,\ldots,a_m$ *and* $b_1,b_2,\ldots,b_m$ *be real numbers such that* $\sum_i a_i=0$ and $\sum_i b_i=1$, $b_i>0$, and consider the curve

$$\Phi\left(t\right)=\sum_{|i|=n}\left(a_1t+b_1\right)^{i_1}\left(a_2t+b_2\right)^{i_2}\cdots\left(a_mt+b_m\right)^{i_m}.$$

We prove that $b=\left(b_1,b_2,\ldots,b_m\right)$ is a local minima for f_m if and only if $b_1=b_2=\cdots=b_m\left(=\frac{1}{m}\right)$.

Indeed, one has

$$\Phi(t) - \Phi(0) \cong \sum_{|i|=n} b_1^{i_1} b_2^{i_2} \cdots b_m^{i_m} \left(\frac{i_1 a_1}{b_1} + \frac{i_2 a_2}{b_2} + \cdots + \frac{i_m a_m}{b_m} \right) t + \cdots$$

$$\cong \binom{n+m-1}{m-1} (b_1 \cdots b_m)^{\binom{n+m-1}{m-1}} \left(\frac{i_1 a_1}{b_1} + \cdots + \frac{i_m a_m}{b_m} \right) t + \cdots .$$

If $b_1 = b_2 = \cdots = b_m \left(= \frac{1}{m}\right)$ then $\Phi(t) - \Phi(0) \cong ct^2 + \cdots ,\ c > 0, \ldots$
If not, we can choose $a_1, a_2, \ldots, a_m$ *such that* $\sum_i \frac{a_i}{b_i} \neq 0, \ldots$
N.B. The possible nullity of some b_i's is not a problem.

(3) the Popoviciu's Theorem given in [7].

3. APPLICATIONS

In this section we apply the previous result to find lower bounds for the sum of reciprocals of multinomial coefficient and for two symmetric functions.

(1) Sum of reciprocals of multinomial coefficient.

Theorem 3.1. *The following inequality holds*

$$\sum_{i_1+i_2+\cdots+i_m=n} \frac{1}{\binom{n}{i_1,\ldots,i_m}} \geq \frac{\binom{n+m-1}{m-1}}{m! \cdot m^n}.$$

Proof. It suffices to integrate each side of the inequality given by the relation (2.2) :

$$f_m(x_1, x_2, \ldots, x_{m-1}, 1 - x_1 - \cdots - x_{m-1}) \geq f_m\left(\tfrac{1}{m}, \tfrac{1}{m}, \ldots, \tfrac{1}{m}\right)$$

over the simplex

$$D = \left\{ x_i,\ i = 1, \ldots, m-1 : x_i \geq 0,\ \sum_{i=1}^{m-1} x_i \leq 1 \right\}.$$

The left hand side gives under the sum the Dirichlet function (or the generalized beta function) and is equal to the reciprocal of a multinomial coefficient. For the right hand side we are led to compute the volume of the simplex D which is equal to $\frac{1}{m!}$. □

(2) An identity due to Sylvester in the 19th century, see [2, Thm 5], states that

Theorem 3.2. *Let $x_1, x_2, \ldots, x_m$ be independent variables. Then, one has in* $\mathbb{R}[x_1, x_2, \ldots, x_m]$

$$\sum_{k_1+\cdots+k_m=n} x_1^{k_1} x_2^{k_2} \ldots x_m^{k_m} = \sum_{i=1}^{m} \frac{x_i^{n+m-1}}{\prod_{j\neq i}(x_i - x_j)}.$$

As corollary of this theorem and Theorem 2.1, one obtains the following lower bound.

Corollary 3.3. *Using the hypothesis of the above theorem, one has*

$$\sum_{i=1}^{m} \frac{x_i^{n+m-1}}{\prod_{j\neq i}(x_i - x_j)} \geq \left(\frac{x_1 + x_2 + \cdots + x_m}{m} \right)^n \binom{n+m-1}{m-1}.$$

(3) The third application is about the symmetric polynomials. We need the following result:

Theorem 3.4 ([2, Cor. 5] and [4, Th. 1]). *Let $x_1, x_2, \dots, x_m$ be elements of unitary commutative ring $\mathcal{A}$ with*

$$\mathrm{s}_k = \sum_{1 \le i_1 < i_2 < \dots < i_k \le m} x_{i_1} x_{i_2} \cdots x_{i_k}, \qquad \text{for } 1 \le k \le m.$$

Then, for each positive integer n, one has

$$\sum x_1^{k_1} \dots x_m^{k_m} = \sum \binom{k_1 + \dots + k_m}{k_1, \dots, k_m} (-1)^{n-k_1-\dots-k_m} \mathrm{s}_1^{k_1} \dots \mathrm{s}_m^{k_m},$$

where the summations are taken over all m-tuples $(k_1, k_2, \dots, k_m)$ of integers $k_j \ge 0$ satisfying the relations $k_1 + k_2 + \dots + k_m = n$ for the left hand side and $k_1 + 2k_2 + \dots + mk_m = n$ for the right hand side.

This theorem and Theorem 2.1, give:

Corollary 3.5. *Using the hypothesis of the last theorem, one has*

$$\frac{1}{\binom{n+m-1}{m-1}} \sum \binom{k_1 + \dots + k_m}{k_1, \dots, k_m} (-1)^{n-k_1-\dots-k_m} \mathrm{s}_1^{k_1} \dots \mathrm{s}_m^{k_m} \ge \left(\frac{\mathrm{s}_1}{m}\right)^n.$$

where the summation is being taken over all m-tuples $(k_1, k_2, \dots, k_m)$ of integers $k_j \ge 0$ satisfying the relation $k_1 + 2k_2 + \dots + mk_m = n$.

REFERENCES

[1] H. ALZER AND J. PEČARIĆ, On an inequality of A.M. Mercer, *Rad Hrvatske Akad. Znan. Umj. Mat*. [467], **11** (1994), 27–30.

[2] H. BELBACHIR AND F. BENCHERIF, Linear recurrent sequences and powers of a square matrix, *Electronic Journal of Combinatorial Number Theory*, A12, **6** (2006).

[3] H. HABER, An elementary inequality, *Internat. J. Math. and Math. Sci.*, **2**(3) (1979), 531–535.

[4] J. MCLAUGHLIN AND B. SURY, Powers of matrix and combinatorial identities, *Electronic Journal of Combinatorial Number Theory*, **A13**(5) (2005).

[5] A. McD. MERCER, A note on a paper by S. Haber, *Internat. J. Math. and Math. Sci.*, **6**(3) (1983), 609–611.

[6] A.McD. MERCER, Polynomials and convex sequence inequalities, *J. Ineq. Pure Appl. Math.*, **6**(1) (2005), Art. 8. [ONLINE: `http://jipam.vu.edu.au/article.php?sid=477`].

[7] T. POPOVICIU, Notes sur les functions convexes d'ordre superieure (V), *Bull. de la sci. Acad. Roumaine*, **22** (1940), 351–356.