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AN ELEMENTARY PROOF OF BLUNDON'S INEQUALITY

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ABSTRACT. In this note, we give an elementary proof of Blundon's Inequality. We make use of a simple auxiliary result, provable by only using the Arithmetic Mean - Geometric Mean Inequality.

Key words and phrases: Blundon's Inequality, Geometric Inequality, Arithmetic-Geometric Mean Inequality.

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For a given triangle ABC we shall consider that A, B, C denote the magnitudes of its angles, and a, b, c denote the lengths of its corresponding sides. Let R, r and s be the circumradius, the inradius and the semi-perimeter of the triangle, respectively. In addition, we will occasionally make use of the symbols $\sum$ (cyclic sum) and $\prod$ (cyclic product), where

$$\sum f(a) = f(a) + f(b) + f(c), \qquad \prod f(a) = f(a)f(b)f(c).$$

In the AMERICAN MATHEMATICAL MONTHLY, W. J. Blundon [1] asked for the proof of the inequality

$$s \leq 2R + (3\sqrt{3} - 4)r$$

which holds in any triangle ABC. The solution given by the editors was in fact a comment made by A. Makowski [3], who refers the reader to [2], where Blundon originally published this inequality, and where he actually proves more, namely that this is the best such inequality in the following sense: if, for the numbers k and h the inequality

$$s \leq kR + hr$$

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is valid in any triangle, with the equality occurring when the triangle is equilateral, then

$$2R + (3\sqrt{3} - 4)r \leq kR + hr.$$

In this note we give a new proof of Blundon's inequality by making use of the following preliminary result:

Lemma 1. *Any positive real numbers $x,\ y,\ z$ such that*

$$x + y + z = xyz$$

satisfy the inequality

$$(x-1)(y-1)(z-1) \leq 6\sqrt{3} - 10.$$

Proof. Since the numbers are positive, from the given condition it follows immediately that $x < xyz \Leftrightarrow yz > 1$, and similarly $xz > 1$ and $yz > 1$, which shows that it is not possible for two of the numbers to be less than or equal to 1 (neither can all the numbers be less than 1). Because if a number is less than 1 and two are greater than 1 the inequality is obviously true (the product from the left-hand side being negative), we still have to consider the case when $x > 1, y > 1, z > 1$. Then the numbers $u = x - 1, v = y - 1$ and $w = z - 1$ are positive and, replacing $x = u + 1$, $y = v + 1$, $z = w + 1$ in the condition from the hypothesis, one gets

$$uvw + uv + uw + vw = 2.$$

By the Arithmetic Mean - Geometric Mean inequality

$$uvw + 3\sqrt[3]{u^2v^2w^2} \leq uvw + uv + uw + vw = 2,$$

and hence for $t = \sqrt[3]{uvw}$ we have

$$t^3 + 3t^2 - 2 \leq 0 \Leftrightarrow (t+1)(t+1+\sqrt{3})(t+1-\sqrt{3}) \leq 0.$$

We conclude that $t \leq \sqrt{3} - 1$ and thus,

$$(x-1)(y-1)(z-1) \leq 6\sqrt{3} - 10.$$

The equality occurs when $x = y = z = \sqrt{3}$. This proves Lemma 1. □

We now proceed to prove Blundon's Inequality.

Theorem 2. *In any triangle ABC, we have that*

$$s \leq 2R + (3\sqrt{3} - 4)r.$$

The equality occurs if and only if ABC is equilateral.

Proof. According to the well-known formulae

$$\cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}, \qquad \cot\frac{B}{2} = \sqrt{\frac{s(s-b)}{(s-c)(s-a)}}, \qquad \cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}},$$

we deduce that

$$\sum \cot\frac{A}{2} = \prod \cot\frac{A}{2} = \frac{s}{r},$$

and

$$\sum \cot\frac{A}{2}\cot\frac{B}{2} = \sum \frac{s}{s-a} = \frac{4R+r}{r}.$$

In this case, by applying Lemma 1 to the positive numbers $x = \cot\frac{A}{2}$, $y = \cot\frac{B}{2}$ and $z = \cot\frac{C}{2}$, it follows that

$$\left(\cot\frac{A}{2} - 1\right)\left(\cot\frac{B}{2} - 1\right)\left(\cot\frac{C}{2} - 1\right) \leq 6\sqrt{3} - 10,$$

and therefore

$$2\prod \cot\frac{A}{2} - \left(\sum \cot\frac{A}{2}\cot\frac{B}{2}\right) \le 6\sqrt{3} - 9.$$

This can be rewritten as

$$\frac{2s}{r} - \frac{4R+r}{r} \le 6\sqrt{3} - 9,$$

and thus

$$s \le 2R + (3\sqrt{3} - 4)r.$$

The equality occurs if and only if $\cot\frac{A}{2} = \cot\frac{B}{2} = \cot\frac{C}{2}$, i.e. when the triangle ABC is equilateral. This completes the proof of Blundon's Inequality. □

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