journal of inequalities in pure and applied mathematics
http://jipam.vu.edu.au
issn: 1443-5756
Volume 8 (2007), Issue 3, Article 80, 8 pp.

ON L^p-ESTIMATES FOR THE TIME DEPENDENT SCHRÖDINGER OPERATOR ON L^2

MOHAMMED HICHEM MORTAD

DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ D'ORAN (ES-SENIA)
B.P. 1524, EL MENOUAR, ORAN. ALGERIA.
mortad@univ-oran.dz

Received 07 July, 2007; accepted 29 August, 2007
Communicated by S.S. Dragomir

ABSTRACT. Let L denote the time-dependent Schrödinger operator in n space variables. We consider a variety of Lebesgue norms for functions u on $\mathbb{R}^{n+1}$, and prove or disprove estimates for such norms of u in terms of the L^2 norms of u and Lu. The results have implications for self-adjointness of operators of the form $L+V$ where V is a multiplication operator. The proofs are based mainly on Strichartz-type inequalities.

Key words and phrases: Schrödinger Equation, Strichartz Estimates and Self-adjointness.

2000 *Mathematics Subject Classification.* Primary 35B45; Secondary 35L10.

1. INTRODUCTION

Let $(x,t)\in\mathbb{R}^{n+1}$ where $n\geq 1$. The Schrödinger equation $\frac{\partial u}{\partial t}=i\triangle_x u$ has been much studied using spectral properties of the self-adjoint operator $\triangle_x$. When a multiplication operator (potential) V is added, it becomes important to determine whether $\triangle_x+V$ is a self-adjoint operator, and there is a vast literature on this question (see e.g. [9]).

One can also, however, regard the operator $L=-i\frac{\partial}{\partial t}-\triangle_x$ as a self-adjoint operator on $L^2(\mathbb{R}^{n+1})$, and that is the point of view taken in this paper. We ask what can be said about the domain of L, more specifically, we ask which L^q spaces, and more generally mixed $L^q_t(L^r_x)$ space, a function u must belong to, given that u is in the domain of L (i.e. u and Lu both belong to $L^2(\mathbb{R}^{n+1})$). We answer this question and, using the Kato-Rellich theorem, deduce sufficient conditions on V for $L+V$ to be self-adjoint.

Our approach is based on the fact that any sufficiently well-behaved function u on $\mathbb{R}^{n+1}$ can be regarded as a solution of the initial value problem (IVP)

$$\begin{cases} -iu_t-\triangle_x u=g(x,t), \\ u(x,\alpha)=f(x) \end{cases} \tag{1.1}$$

I would like to express all my gratitude to my ex-Ph.D.-supervisor Professor Alexander M. Davie for his helpful comments, especially in the counterexamples section.

227-07

where $\alpha \in \mathbb{R}$, $f(x) = u(x, \alpha)$ and $g = Lu$.

To apply this, we will use estimates for u based on given bounds for f and g. A number of such estimates are known and generally called Strichartz inequalities, after [12] which obtained such an L^q bound for u. This has since been generalized to give inequalities for mixed norms [13, 4]. The specific inequalities we use concern the case $g = 0$ of (1.1) and give bounds for u in terms of $\|f\|_{L^2(\mathbb{R}^n)}$ - see (3.2) below. The precise range of mixed $L^q_t(L^r_x)$ norms for which the bound (3.2) holds is known as a result of [13, 4] and the counterexample in [6].

In Section 2 we prove a special case of our main theorem, namely a bound for u in $L^\infty_t(L^2_x)$, which does not require Strichartz estimates, only elementary arguments using the Fourier transform. The main theorem, giving $L^q_t(L^r_x)$ bounds for the largest possible set of (q, r) pairs, is proved in Section 3. In fact, we prove a somewhat stronger bound, in a smaller space $\mathcal{L}_{2,q,r}$ defined below. The fact that the set of pairs (q, r) covered by Theorem 3.1 is the largest possible is shown in Section 4.

Some results on a similar question for the wave operator can be found in [7]. For Strichartz-type inequalities for the wave operator, see e.g. [11, 12, 2, 3, 4].

We assume notions and definitions about the Fourier Transform and unbounded operators and for a reference one may consult [8], [5] or [10]. We also use on several occasions the well-known Duhamel principle for the Schrödinger equation (see e.g. [1]).

Notation. The symbol $\hat{u}$ stands for the Fourier transform of u in the space (x) variable while the inverse Fourier transform will be denoted either by $\mathcal{F}^{-1}u$ or $\check{u}$.

We denote by $C_0^\infty(\mathbb{R}^{n+1})$ the space of infinitely differentiable functions with compact support.

We denote by $\mathbb{R}^+$ the set of all positive real numbers together with $+\infty$.

For $1 \le p \le \infty$, $\|\cdot\|_p$ is the usual L^p-norm whereas $\|\cdot\|_{L^p_t(L^q_x)}$ stands for the mixed spacetime Lebesgue norm defined as follows

$$\|u\|_{L^q_t(L^r_x)} = \left(\int_{\mathbb{R}} \|u(t)\|^q_{L^r_x} dt\right)^{\frac{1}{q}}.$$

We also define some modified mixed norms. First we define, for any integer k,

$$\|u\|_{L^q_{t,k}(L^r_x)} = \left(\int_k^{k+1} \|u(t)\|^q_{L^r_x} dt\right)^{\frac{1}{q}},$$

and then

$$\|u\|_{\mathcal{L}_{p,q,r}} = \left(\sum_{k\in\mathbb{Z}} \|u\|^p_{L^q_{t,k}(L^r_x)}\right)^{\frac{1}{p}}.$$

We note that $\|u\|_{\mathcal{L}_{p,q_1,r}} \ge \|u\|_{\mathcal{L}_{p,q_2,r}}$ if $q_1 \ge q_2$, and that $\|u\|_{L^q_t(L^r_x)} \le \|u\|_{\mathcal{L}_{p,q,r}}$ if $q \ge p$.

Finally we define

$$M^n_L = \{f \in L^2(\mathbb{R}^{n+1}) : Lf \in L^2(\mathbb{R}^{n+1})\},$$

where L is defined as in the abstract and where the derivative is taken in the distributional sense. We note that $M^n_L = \mathcal{D}(L)$, the domain of L, and also that $C_0^\infty(\mathbb{R}^{n+1})$ is dense in M^n_L in the graph norm $\|u\|_{L^2(\mathbb{R}_{n+1})} + \|Lu\|_{L^2(\mathbb{R}_{n+1})}$.

2. $L_t^\infty(L_x^2)$ ESTIMATES.

Before stating the first result, we are going to prepare the ground for it. Take the Fourier transform of the IVP (1.1) in the space variable to get

$$\begin{cases} -i\hat{u}_t + \eta^2\hat{u} = \hat{g}(\eta, t), \\ \hat{u}(\eta, \alpha) = \hat{f}(\eta) \end{cases}$$

which has the following solution (valid for all $t \in \mathbb{R}$):

$$\hat{u}(\eta, t) = \hat{f}(\eta)e^{-i\eta^2 t} + i\int_\alpha^t e^{-i\eta^2(t-s)}\hat{g}(\eta, s)ds, \tag{2.1}$$

where $\eta \in \mathbb{R}^n$.

Duhamel's principle gives an alternative way of writing the part of the solution depending on g. Taking the case $f = 0$, the solution of (1.1) can be written as

$$u(x, t) = i\int_\alpha^t u_s(x, t)ds, \tag{2.2}$$

where u_s is the solution of

$$\begin{cases} Lu_s = 0, & t > s, \\ u_s(x, s) = g(x, s). \end{cases}$$

Now we state a result which we can prove using (2.1). In the next section we prove a more general result using Strichartz inequalities and Duhamel's principle (2.2).

Proposition 2.1. *For all $a > 0$, there exists $b > 0$ such that*

$$\|u\|_{\mathcal{L}_{2,\infty,2}} \leq a\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} + b\|u\|^2_{L^2(\mathbb{R}^{n+1})}$$

for all $u \in M_L^n$.

Proof. We prove the result for $u \in C_0^\infty(\mathbb{R}^{n+1})$ and a density argument allows us to deduce it for $u \in M_L^n$.

We use the fact that any such u is, for any $\alpha \in \mathbb{R}$, the unique solution of (1.1), where $f(x) = u(x, \alpha)$ and $g = Lu$, and therefore satisfies (2.1).

Let $k \in \mathbb{Z}$ and let t and α be such that $k \leq t \leq k+1$ and $k \leq \alpha \leq k+1$. Squaring (2.1), integrating with respect to η in $\mathbb{R}^n$, and using Cauchy-Schwarz (and the fact that $|t - \alpha| \leq 1$), we obtain

$$\|\hat{u}(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} \leq 2\int_{\mathbb{R}^n} |\hat{u}(\eta, \alpha)|^2 d\eta + 2\int_{\mathbb{R}^n}\int_\alpha^t |\hat{g}(\eta, s)|^2 ds d\eta. \tag{2.3}$$

Now integrating against α in $[k, k+1]$ allows us to say that

$$\|u(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} \leq 2\int_k^{k+1}\int_{\mathbb{R}^n} |\hat{u}(\eta, \alpha)|^2 d\eta d\alpha + 2\int_k^{k+1}\int_{\mathbb{R}^n} |\hat{g}(\eta, s)|^2 d\eta ds.$$

Now take the essential supremum of both sides in t over $[k, k+1]$, then sum in k over $\mathbb{Z}$ to get (recalling that $g = Lu$)

$$\sum_{k=-\infty}^{\infty} \operatorname{ess}\sup_{k \leq t \leq k+1} \|u(\cdot, t)\|^2_{L^2(\mathbb{R}^n)} \leq 2\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} + 2\|u\|^2_{L^2(\mathbb{R}^{n+1})}.$$

Finally to get an arbitrarily small constant in the Lu term we use a scaling argument: let m be a positive integer and let $v(x, t) = u(mx, m^2t)$. Then we find

$$\|v\|_{L^2(\mathbb{R}^{n+1})} = m^{-1-n/2}\|u\|_{L^2(\mathbb{R}^{n+1})}$$

and

$$\|Lv\|_{L^2(\mathbb{R}^{n+1})} = m^{1-n/2}\|Lu\|_{L^2(\mathbb{R}^{n+1})}.$$

Also,

$$\|v(\cdot,t)\|_{L^2(\mathbb{R}^n)} = m^{-n/2}\|u(\cdot,m^2t)\|_{L^2(\mathbb{R}^n)}$$

and so

$$\begin{aligned}\sup_{k\le t\le k+1}\|v(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &= m^{-n}\sup_{m^2k\le t\le m^2(k+1)}\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}\\ &\le m^{-n}\sum_{j=m^2k}^{m^2(k+1)-1}\sup_{j\le t\le j+1}\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Summing over k gives

$$\begin{aligned}\|v\|^2_{\mathcal{L}_{2,\infty,2}} &\le m^{-n}\|u\|^2_{\mathcal{L}_{2,\infty,2}}\\ &\le m^{-n}\left(2\|Lu\|^2_{L^2(\mathbb{R}^{n+1})} + 2\|u\|^2_{L^2(\mathbb{R}^{n+1})}\right)\\ &\le 2m^{-2}\|Lv\|^2_{L^2(\mathbb{R}^{n+1})} + 2m^2\|v\|^2_{L^2(\mathbb{R}^{n+1})}\end{aligned}$$

and choosing m so that $2m^{-2} < a$ completes the proof. □

Now we recall the Kato-Rellich theorem which states that if L is a self-adjoint operator on a Hilbert space and V is a symmetric operator defined on $\mathcal{D}(L)$, and if there are positive constants $a < 1$ and b such that $\|Vu\| \le a\|Lu\| + b\|u\|$ for all $u \in \mathcal{D}(L)$, then $L+V$ is self-adjoint on $\mathcal{D}(L)$ (see [9]).

Corollary 2.2. *Let V be a real-valued function in $\mathcal{L}_{\infty,2,\infty}$. Then $L+V$ is self-adjoint on $\mathcal{D}(L) = M^n_L$.*

Proof. One can easily check that

$$\|Vu\|_{L^2(\mathbb{R}^{n+1})} \le \|V\|_{\mathcal{L}_{\infty,2,\infty}}\|u\|_{\mathcal{L}_{2,\infty,2}}.$$

Choose $a < \|V\|^{-1}_{\mathcal{L}_{\infty,2,\infty}}$ and then Proposition 2.1 shows that $L+V$ satisfies the hypothesis of the Kato-Rellich theorem. □

In particular, it follows that $L+V$ is self-adjoint whenever $V \in L^2_t(L^\infty_x)$.

3. $L^q_t(L^r_x)$ ESTIMATES.

Now we come to the main theorem in this paper, which depends on the following Strichartz-type inequality. Suppose $n \ge 1$ and q and r are positive real numbers (possibly infinite) such that $q \ge 2$ and

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{2}. \tag{3.1}$$

When $n = 2$ we exclude the case $q = 2$, $r = \infty$. Then there is a constant C such that if $f \in L^2(\mathbb{R}^n)$ and $g = 0$, the solution u of (1.1) satisfies

$$\|u\|_{L^q_t(L^r_x)} \le C\|f\|_{L^2(\mathbb{R}^n)}. \tag{3.2}$$

This result can be found in [13] for $q > 2$; the more difficult 'end-point' case where $q = 2$, $n \ge 3$ is treated in [4]. That (3.2) fails in the exceptional case $n = 2, q = 2, r = \infty$ is shown in [6].

For $n \geq 1$ we define a region $\Omega_n \in \mathbb{R}^+ \times \mathbb{R}^+$ as follows: for $n \neq 2$,

$$\Omega_n = \left\{ (q,r) \in \mathbb{R}^+ \times \mathbb{R}^+ : \frac{2}{q} + \frac{n}{r} \geq \frac{n}{2},\ q \geq 2,\ r \geq 2 \right\} \tag{3.3}$$

and for $n = 2$, Ω_2 is defined by the same expression, with the omission of the point $(2, \infty)$.

The sets Ω_n are probably most easily visualized in the $(\frac{1}{q}, \frac{1}{r})$-plane. Then Ω_1 is a quadrilateral with vertices $(\frac{1}{4}, 0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$ and for $n \geq 2$, Ω_n is a triangle with vertices $(\frac{1}{2}, \frac{n-2}{2n}), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$, the point $(\frac{1}{2}, 0)$ being excluded in the case $n = 2$.

Theorem 3.1. *Let $n \geq 1$, and let $(q,r) \in \Omega_n$. Then for all $a > 0$, there exists $b > 0$ such that*

$$\|u\|_{\mathcal{L}_{2,q,r}} \leq a \|Lu\|_{L^2(\mathbb{R}^{n+1})} + b \|u\|_{L^2(\mathbb{R}^{n+1})} \tag{3.4}$$

for all $u \in M_L^n$.

Proof. By the inclusion $\mathcal{L}_{2,q_1,r} \subseteq \mathcal{L}_{2,q_2,r}$, when $q_1 \geq q_2$ it suffices to treat the case where $\frac{2}{q} + \frac{n}{r} = \frac{n}{2}$, for which (3.2) holds.

Let $k \in \mathbb{Z}$ and let $\alpha \in [k, k+1]$. As in the proof of Proposition 2.1 we use the fact that u is the solution of (1.1) with $f = u(\cdot, \alpha)$ and $g = Lu$. Now we split u into two parts $u = u_1 + u_2$, where u_1, u_2 are the solutions of

$$\begin{cases} Lu_1 = g, \\ u_1(x,\alpha) = 0, \end{cases} \qquad \begin{cases} Lu_2 = 0, \\ u_2(x,\alpha) = f. \end{cases}$$

The estimate for u_2 is deduced from (3.2):

$$\|u_2\|_{L_t^q(L_x^r)} \leq C \|f\|_{L^2(\mathbb{R}^n)} \leq C \|u(\cdot,\alpha)\|_{L^2(\mathbb{R}^n)}. \tag{3.5}$$

For u_1 we apply (2.2) to obtain

$$u_1(x,t) = i \int_\alpha^t u_s(x,t)\,ds, \tag{3.6}$$

from which we deduce

$$\|u_1(\cdot,t)\|_{L^r(\mathbb{R}^n)} \leq \int_k^{k+1} \|u_s(\cdot,t)\|_{L^r(\mathbb{R}^n)}\,ds$$

for $t \in [k, k+1]$, and hence

$$\begin{aligned} \|u_1\|_{L_{t,k}^q(L_x^r)} &\leq \int_k^{k+1} \|u_s\|_{L_t^q(L_x^r)}\,ds \\ &\leq C \int_k^{k+1} \|g(\cdot,s)\|_{L^2(\mathbb{R}^n)}\,ds \\ &\leq C \|g\|_{L^2(\mathbb{R}^n \times [k,k+1])}. \end{aligned}$$

Combining this with (3.5) we have

$$\|u\|_{L_{t,k}^q(L_x^r)}^2 \leq 2C^2 \|u(\cdot,\alpha)\|_{L^2(\mathbb{R}^n)}^2 + 2C^2 \|Lu\|_{L^2(\mathbb{R}^n \times [k,k+1])}^2.$$

Integrating w.r.t. α from k to $k+1$ gives

$$\|u\|_{L_{t,k}^q(L_x^r)}^2 \leq 2C^2 \|u\|_{L^2(\mathbb{R}^n \times [k,k+1])}^2 + 2C^2 \|Lu\|_{L^2(\mathbb{R}^n \times [k,k+1])}^2.$$

Summing over k, we obtain

$$\|u\|_{\mathcal{L}_{2,q,r}}^2 \leq 2C^2 \|u\|_{L^2(\mathbb{R}^{n+1})} + 2C^2 \|Lu\|_{L^2(\mathbb{R}^{n+1})},$$

and the proof is completed by a similar scaling argument to that used in Proposition 2.1. □

Using the inclusion $\mathcal{L}_{2,q,r} \subseteq L^q_t(L^r_x)$ for $q \geq 2$ we deduce

Corollary 3.2. *Let $n \geq 1$, and let $(q,r) \in \Omega_n$. Then for all $a > 0$, there exists $b > 0$ such that*

$$\|u\|_{L^q_t(L^r_x)} \leq a\|Lu\|_{L^2(\mathbb{R}^{n+1})} + b\|u\|_{L^2(\mathbb{R}^{n+1})} \tag{3.7}$$

for all $u \in M^n_L$.

In particular, we get such a bound for $\|u\|_{L^q(\mathbb{R}^{n+1})}$ whenever $2 \leq q \leq (2n+4)/n$.

By applying the Kato-Rellich theorem we can deduce a generalization of Corollary 2.2 from Theorem 3.1. We first define

$$\Omega^*_n = \left\{(p,s) \in \mathbb{R}^+ \times \mathbb{R}^+ : \ \frac{2}{p} + \frac{n}{s} \leq 1, \ p \geq 2, \ s \geq 2\right\} \tag{3.8}$$

for $n \neq 2$, and for $n = 2$, Ω_2 is defined by the same expression, with the omission of the point $(2,\infty)$.

Corollary 3.3. *Let $n \geq 1$ and let $(p,s) \in \Omega^*_n$. Let V be a real-valued function belonging to $\mathcal{L}_{\infty,p,s}$. Then $L+V$ is self-adjoint on M^n_L.*

Proof. Let $q = \frac{2p}{p-2}$ and $r = \frac{2s}{s-2}$. Then $(q,r) \in \Omega_n$ and the conclusion (3.4) of Theorem 3.1 applies. Now we have

$$\begin{aligned}\int_k^{k+1} \|Vu(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} &\leq \int_k^{k+1} \|u(\cdot,t)\|^2_{L^r(\mathbb{R}^n)} \|V(\cdot,t)\|^2_{L^s(\mathbb{R}^n)} \\ &\leq \|u\|^2_{L^q_{t,k}(L^r_x)} \|V\|^2_{L^p_{t,k}(L^s_x)}\end{aligned}$$

and summation over k gives

$$\|Vu\|_{L^2(\mathbb{R}^{n+1})} \leq \|u\|_{\mathcal{L}_{2,q,r}} \|V\|_{\mathcal{L}_{\infty,p,s}}.$$

Then, using (3.4), the result follows in the same way as Corollary 2.2. ☐

It follows from Corollary 3.3 that $L+V$ is self-adjoint whenever $V \in L^p_t(L^s_x)$ for $(p,s) \in \Omega^*_n$. Taking the case $s = p$, we find that $L+V$ is self-adjoint if $V \in L^p(\mathbb{R}^{n+1})$ for some $p \geq n+2$.

4. COUNTEREXAMPLES

Now we show that Theorem 3.1 is sharp, as far as the allowed set of q, r is concerned.

Proposition 4.1. *Let $n \geq 1$ and let q and r be positive real numbers, possibly infinite, such that $(q,r) \notin \Omega_n$. Then there are no constants a and b such that (3.7) holds for all $u \in M^n_L$.*

Proof. For (q,r) to fail to be in Ω_n one of the following three possibilities must occur: (i) $q < 2$ or $r < 2$; (ii) $\frac{2}{q} + \frac{n}{r} < \frac{n}{2}$; (iii) $n = 2$, $q = 2$ and $r = \infty$. We consider these cases in turn.

(i) If $q < 2$, choose a sequence $(\beta_k)_{k\in\mathbb{Z}}$ which is in l^2 but not in l^q. Let $\phi(x,t)$ be a smooth function of compact support on $\mathbb{R}^{n+1}$ which vanishes for t outside $[0,1]$, and let $u(x,t) = \sum_{k\in\mathbb{Z}} \beta_k \phi(x, t-k)$. Then $u \in M^n_L$, but $u \notin L^q_t(L^r_x)$ for any r.

The case $r < 2$ can be treated similarly. We chose a sequence β_k which is in l^2 but not l^r, and a smooth ϕ which vanishes for x_1 outside $[0,1]$, then set $u(x,t) = \sum_{k\in\mathbb{Z}} \beta_k \phi(x - ke_1, t)$, where e_1 is the unit vector $(1,0,\ldots,0)$ in $\mathbb{R}^n$. Then $u \in M^n_L$, but $u \notin L^q_t(L^r_x)$ for any q.

(ii) In this case we use the scaling argument which shows that the Strichartz estimates fail, together with a cutoff to ensure u and Lu are in L^2.

We start with a non-zero $f \in L^2(\mathbb{R}^n)$, and let u be the solution of (1.1) with $\alpha = 0$ and $g = 0$. (An explicit example would be $f(x) = e^{-|x|^2}$ and then $u(x,t) = (1+4it)^{-n/2} e^{-|x|^2/(1+4it)}$).

Choose a smooth function ϕ on $\mathbb{R}$ such that $\phi(0) \neq 0$ and such that ϕ and ϕ' are in L^2. Then for $\lambda > 0$ define

$$v_\lambda(x,t) = \lambda^{n/2} u(\lambda x, \lambda^2 t)\phi(t).$$

Then (using $Lu = 0$) we find $Lv(x,t) = -i\lambda^{n/2}u(\lambda x, \lambda^2 t)\phi'(t)$. We calculate $\|v_\lambda\|_{L^2(\mathbb{R}^{n+1})} = \|f\|_{L^2(\mathbb{R}^n)}\|\phi\|_{L^2}$ and $\|Lv_\lambda\|_{L^2(\mathbb{R}^{n+1})} = \|f\|_{L^2(\mathbb{R}^n)}\|\phi'\|_{L^2}$. Also

$$\|v_\lambda\|_{L^q_t(L^r_x)} = \lambda^\beta \left\{ \int_{\mathbb{R}} \|u(\cdot,t)\|^q_{L^r(\mathbb{R}^n)} |\phi(\lambda^{-2}t)|^q dt \right\}^{\frac{1}{q}},$$

where $\beta = \frac{n}{2} - \frac{n}{r} - \frac{2}{q} > 0$. So $\lambda^{-\beta}\|v_\lambda\|_{L^q_t(L^r_x)} \to |\phi(0)|\|u\|_{L^q_t(L^r_x)}$ (note that the norm on the right may be infinite) and hence $\|v_\lambda\|_{L^q_t(L^r_x)}$ tends to ∞ as $\lambda \to \infty$, completing the proof.

(iii) This exceptional case we treat in a similar fashion to (ii), but we need the result from [6], that the Strichartz inequality fails in this case. We start by fixing a smooth function ϕ on $\mathbb{R}$ such that $\phi = 1$ on $[-1,1]$ and ϕ and ϕ' are in L^2.

Now let $M > 0$ be given and we use [6] to find $f \in L^2(\mathbb{R}^2)$ with $\|f\|_{L^2(\mathbb{R}^2)} = 1$ such that the solution u of (1.1) with $\alpha = 0$ and $g = 0$ satisfies $\|u\|_{L^2_t(L^\infty_x)} > M$. Then we can find $R > 0$ so that $\int_{-R}^R \|u(\cdot,t)\|^2_{L^\infty(\mathbb{R}^2)}dt > M^2$. Let $\lambda = R^{1/2}$ and define $v(x,t) = \lambda^{n/2}u(\lambda x, \lambda^2 t)\phi(t)$. Then $\|v\|_{L^2(\mathbb{R}^3)} = \|\phi\|_{L^2}$, $\|Lv\|_{L^2(\mathbb{R}^3)} = \|\phi'\|_{L^2}$ and

$$\|v\|^2_{L^2_t(L^\infty_x)} \geq \int_{-1}^1 \|v(\cdot,t)\|^2_{L^\infty(\mathbb{R}^2)}dt > M^2,$$

which completes the proof, since M is arbitrary. □

We remark that [6] also gives an example of $f \in L^2(\mathbb{R}^2)$ such that $u \notin L^2_t(BMO_x)$ and the argument of part (iii) can then be applied to show that no inequality

$$\|u\|_{L^2_t(BMO_x)} \leq a\|Lu\|_{L^2(\mathbb{R}^3)} + b\|u\|_{L^2(\mathbb{R}^3)}$$

can hold.

5. QUESTION

We saw as a result of Corollary 3.3 that if $(p,s) \in \Omega^*$, then $L + V$ is self-adjoint on M^n_L whenever $V \in L^p_t(L^s_x)$. One can ask whether this can be extended to a larger range of (p,s) with $p, s \geq 2$. If one asks whether $L + V$ is defined on M^n_L, then we would require a bound $\|Vu\|_{L^2(\mathbb{R}^{n+1})} \leq a\|Lu\|_{L^2(\mathbb{R}^{n+1})} + b\|u\|$ to hold for all $u \in M^n_L$. If such a bound is to hold for all $V \in L^p_t(L^s_x)$, then, in fact, we require (3.7) to hold for $q = \frac{2p}{p-2}$ and $r = \frac{2s}{s-2}$, which we know cannot hold unless $(p,s) \in \Omega^*$.

One can instead ask for $L + V$, defined on say $C_0^\infty(\mathbb{R}^{n+1})$, to be essentially self-adjoint. This is equivalent to saying that the only (distribution) solution in $L^2(\mathbb{R}^{n+1})$ of the PDE

$$-iu_t - \triangle_x u + Vu = \pm iu$$

is $u = 0$ (see e.g. [8]).

We do not know if there are any values of (p,s) not in Ω^*_n such that this holds for all $V \in L^p_t(L^s_x)$. The analogous question for the Laplacian is extensively discussed in [9].

REFERENCES

[1] L.C. EVANS, *Partial Differential Equations*, American Mathematical Society (G.S.M. Vol. 19), 1998.

[2] J. GINIBRE AND G. VELO, Generalized Strichartz inequalities for the wave equation, *J. of Functional Analysis*, **133** (1995), 50–68.

[3] J. HARMSE, On Lebesgue space estimates for the wave equation, *Indiana Univ. Math. J.*, **39**(1) (1990) 229–248.

[4] M. KEEL AND T. TAO, Endpoint Strichartz estimates, *Amer. J. of Math.*, **120** (1998), 955–980.

[5] E.H. LIEB AND M. LOSS, *Analysis*, American Mathematical Society (G.S.M. Vol. 14), Second Edition 2001.

[6] S.J. MONTGOMERY-SMITH, Time decay for the bounded mean oscillation of solutions of the Schrödinger and wave equations, *Duke Math. J.*, **91**(2) (1998), 393–408.

[7] M.H. MORTAD, Self-adjointness of the perturbed wave operator on $L^2(\mathbf{R}^\mathbf{n})$, *Proc. Amer. Math. Soc.*, **133**(2) (2005), 455–464.

[8] M. REED AND B. SIMON, *Methods of Modern Mathematical Physics*, Vol. 1, "Functional Analysis," Academic Press, 1972.

[9] M. REED AND B. SIMON, *Methods of Modern Mathematical Physics*, Vol.2, "Fourier Analysis, Self-Adjointness," Academic Press, 1975.

[10] E.M. STEIN AND G. WEISS, *Introduction to Fourier Analysis on Euclidean Spaces*, (PMS; **32**) Princeton University Press, 1971.

[11] R.S. STRICHARTZ, A priori estimates for the wave equation and some applications, *J. of Functional Analysis*, **5** (1970), 218–235.

[12] R.S. STRICHARTZ, Restrictions of Fourier transforms to quadratic surfaces and decay of solutions of wave equations, *Duke Mathematical Journal*, **44**(3) (1977), 705–714.

[13] K. YAJIMA, Existence of solutions for Schrödinger evolution equations, *Comm. Math. Phys.*, **110** (1987), 415–426.