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SOME ESTIMATES ON THE WEAKLY CONVERGENT SEQUENCE COEFFICIENT IN BANACH SPACES

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Abstract

In this paper, we study the weakly convergent sequence coefficient and obtain its estimates for some parameters in Banach spaces, which give some sufficient conditions for a Banach space to have normal structure.

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Contents

1. Introduction

A Banach space X said to have *(weak) normal structure* provided for every (weakly compact) closed bounded convex subset C of X with $\operatorname{diam}(C) > 0$, contains a nondiametral point, i.e., there exists $x_0 \in C$ such that $\sup\{\|x - x_0\| : x \in C\} < \operatorname{diam}(C)$. It is clear that normal structure and weak normal structure coincides when X is reflexive.

The weakly convergent sequence coefficient $WCS(X)$, a measure of weak normal structure, was introduced by Bynum in [3] as the following.

Definition 1.1. *The weakly convergent sequence coefficient of X is defined by*

$$WCS(X) = \inf\left\{\frac{\operatorname{diam}_a(\{x_n\})}{r_a(\{x_n\})} : \{x_n\} \text{ is a weakly convergent sequence}\right\}, \tag{1.1}$$

where $\operatorname{diam}_a(\{x_n\}) = \limsup_{k\to\infty}\{\|x_n - x_m\| : n, m \geq k\}$ *is the asymptotic diameter of* $\{x_n\}$ *and* $r_a(\{x_n\}) = \inf\{\limsup_{n\to\infty}\|x_n - y\| : y \in \overline{co}(\{x_n\})$ *is the asymptotic radius of* $\{x_n\}$.

One of the equivalent forms of $WCS(X)$ is

$$WCS(X) = \inf\left\{\lim_{n,m,n\neq m}\|x_n - x_m\| : x_n \xrightarrow{w} 0, \|x_n\| = 1 \text{ and } \lim_{n,m,n\neq m}\|x_n - x_m\| \text{ exists}\right\}.$$

Obviously, $1 \le WCS(X) \le 2$, and it is well known that $WCS(X) > 1$ implies that X has a weak normal structure.

The constant $R(a, X)$, which is a generalized García-Falset coefficient [10], was introduced by Domínguez [7] as: For a given real number $a > 0$,

$$R(a, X) = \sup \left\{ \liminf_{n\to\infty} \|x + x_n\| \right\}, \tag{1.2}$$

where the supremum is taken over all $x \in X$ with $\|x\| \le a$ and all weakly null sequences $\{x_n\} \subseteq B_X$ such that

$$\lim_{n,m,n\neq m} \|x_n - x_m\| \le 1. \tag{1.3}$$

We shall assume throughout this paper that B_X and S_X to denote the unit ball and unit sphere of X, respectively. $x_n \xrightarrow{w} x$ stands for weak convergence of sequence $\{x_n\}$ in X to a point x in X.

2. Main Results

The *James constant*, or the *nonsquare constant*, was introduced by Gao and Lau in [8] as

$$\begin{aligned} J(X) &= \sup\{\|x+y\| \wedge \|x-y\| : x, y \in S_X\} \\ &= \sup\{\|x+y\| \wedge \|x-y\| : x, y \in B_X\}. \end{aligned}$$

A relation between the constant $R(1, X)$ and the James constant $J(X)$ can be found in [6, 12]:

$$R(1, X) \leq J(X).$$

We now state an inequality between the James constant $J(X)$ and the weakly convergent sequence coefficient $WCS(X)$.

Theorem 2.1. *Let X be a Banach space with the James constant $J(X)$. Then*

$$WCS(X) \geq \frac{J(X)+1}{(J(X))^2}. \tag{2.1}$$

Proof. If $J(X) = 2$, it suffices to note that $WCS(X) \geq 1$. Thus our estimate is a trivial one.

If $J(X) < 2$, then X is reflexive. Let $\{x_n\}$ be a weakly null sequence in S_X. Assume that $d = \lim_{n,m,n\neq m} \|x_n - x_m\|$ exists and consider a normalized functional sequence $\{x_n^*\}$ such that $x_n^*(x_n) = 1$. Note that the reflexivity of X guarantees, by passing through the subsequence, that there exists $x^* \in X^*$ such that $x_n^* \xrightarrow{w} x^*$. Let $0 < \epsilon < 1$ and choose N large enough so that $|x^*(x_N)| < \epsilon/2$ and

$$d - \epsilon < \|x_N - x_m\| < d + \epsilon$$

for all $m > N$. Note that

$$\lim_{n,m,n\neq m} \left\| \frac{x_n - x_m}{d+\epsilon} \right\| \leq 1 \qquad \text{and} \qquad \left\| \frac{x_N}{d+\epsilon} \right\| \leq 1.$$

Then by the definition of $R(1,X)$, we can choose $M > N$ large enough such that

$$\left\| \frac{x_N + x_M}{d+\epsilon} \right\| \leq R(1,X) + \epsilon \leq J(X) + \epsilon, \qquad |(x_M^* - x^*)(x_N)| < \epsilon/2,$$

and $|x_N^*(x_M)| < \epsilon$. Hence

$$|x_M^*(x_N)| \leq |(x_M^* - x^*)(x_N))| + |x^*(x_N)| < \epsilon.$$

Put $\alpha = J(X)$,

$$x = \frac{x_N - x_M}{d+\epsilon}, \qquad \text{and} \qquad y = \frac{x_N + x_M}{(d+\epsilon)(\alpha+\epsilon)}.$$

It follows that $\|x\| \leq 1$, $\|y\| \leq 1$, and also that

$$\begin{aligned} \|x+y\| &= \frac{1}{(d+\epsilon)(\alpha+\epsilon)} \|(\alpha+1+\epsilon)x_N - (\alpha-1+\epsilon)x_M\| \\ &\geq \frac{1}{(d+\epsilon)(\alpha+\epsilon)} ((\alpha+1+\epsilon)x_N^*(x_N) - (\alpha-1+\epsilon)x_N^*(x_M)) \\ &\geq \frac{\alpha+1-\epsilon}{(d+\epsilon)(\alpha+\epsilon)}, \end{aligned}$$

$$
\begin{aligned}
\|x-y\| &= \frac{1}{(d+\epsilon)(\alpha+\epsilon)}\big\|(\alpha+1+\epsilon)x_M-(\alpha-1+\epsilon)x_N\big\| \\
&\geq \frac{1}{(d+\epsilon)(\alpha+\epsilon)}\big((\alpha+1+\epsilon)x_M^*(x_M)-(\alpha-1+\epsilon)x_M^*(x_N)\big) \\
&\geq \frac{\alpha+1-\epsilon}{(d+\epsilon)(\alpha+\epsilon)}.
\end{aligned}
$$

Thus, from the definition of the James constant,

$$
J(X) \geq \frac{\alpha+1-\epsilon}{(d+\epsilon)(\alpha+\epsilon)} = \frac{J(X)+1-\epsilon}{(d+\epsilon)(J(X)+\epsilon)}.
$$

Letting $\epsilon \to 0$, we get

$$
d \geq \frac{J(X)+1}{(J(X))^2}.
$$

Since the sequence $\{x_n\}$ is arbitrary, we get the inequality (2.1). □

As an application of Theorem 2.1, we can obtain a sufficient condition for X to have normal structure in terms of the James constant.

Corollary 2.2 ([4, Theorem 2.1]). *Let X be a Banach space with $J(X) < (1+\sqrt{5})/2$. Then X has normal structure.*

The *modulus of smoothness* [14] of X is the function $\rho_X(\tau)$ defined by

$$
\rho_X(\tau) = \sup\left\{\frac{\|x+\tau y\|+\|x-\tau y\|}{2} - 1 : x, y \in S_X\right\}.
$$

It is readily seen that for any $x, y \in S_X$,

$$\|x \pm y\| \leq \|x \pm \tau y\| + (1 - \tau) \qquad (0 < \tau \leq 1),$$

which implies that $J(X) \leq \rho_X(\tau) + 2 - \tau$.

In [2], Baronti et al. introduced a constant $A_2(X)$, which is defined by

$$A_2(X) = \rho_X(1) + 1 = \sup\left\{\frac{\|x + y\| + \|x - y\|}{2} : x, y \in S_X\right\}.$$

It is worth noting that $A_2(X) = A_2(X^*)$.

We now state an inequality between the modulus of smoothness $\rho_X(\tau)$ and the weakly convergent sequence coefficient $WCS(X)$.

Theorem 2.3. *Let X be a Banach space with the modulus of smoothness $\rho_X(\tau)$. Then for any $0 < \tau \leq 1$,*

$$WCS(X) \geq \frac{\rho_X(\tau) + 2}{(\rho_X(\tau) + 1)(\rho_X(\tau) - \tau + 2)}. \tag{2.2}$$

Proof. Let $0 < \tau \leq 1$. If $\rho_X(\tau) = \tau$, it suffices to note that

$$\frac{\rho_X(\tau) + 2}{(\rho_X(\tau) + 1)(\rho_X(\tau) - \tau + 2)} = \frac{\tau + 2}{2(\tau + 1)} \leq 1.$$

Thus our estimate is a trivial one.

If $\rho_X(\tau) < \tau$, then X is reflexive. Let $\{x_n\}$ be a weakly null sequence in S_X. Assume that $d = \lim n, m, n \neq m\|x_n - x_m\|$ exists and consider a normalized functional sequence $\{x_n^*\}$ such that $x_n^*(x_n) = 1$. Note that the reflexivity of X guarantees that there exists $x^* \in X^*$ such that $x_n^* \xrightarrow{w} x^*$. Let $\epsilon > 0$ and x_M, x_N, x and y selected as in Theorem 2.1. Similarly, we get

$$\|x \pm \tau y\| \geq \frac{\alpha(\tau) + \tau - \epsilon}{(d+\epsilon)(\alpha(\tau)+\epsilon)},$$

where $\alpha(\tau) = \rho_X(\tau) + 2 - \tau$. Then by the definition of $\rho_X(\tau)$, we obtain

$$\rho_X(\tau) \geq \frac{\alpha(\tau) + \tau - \epsilon}{(d+\epsilon)(\alpha(\tau)+\epsilon)} - 1.$$

Letting $\epsilon \to 0$,

$$\rho_X(\tau) + 1 \geq \frac{\alpha(\tau) + \tau}{d\alpha(\tau)} = \frac{\rho_X(\tau) + 2}{d(\rho_X(\tau) - \tau + 2)},$$

which gives that

$$d \geq \frac{\rho_X(\tau) + 2}{(\rho_X(\tau) + 1)(\rho_X(\tau) - \tau + 2)}.$$

Since the sequence $\{x_n\}$ is arbitrary, we get the inequality (2.2). □

It is known that if $\rho_X(\tau) < \tau/2$ for some $\tau > 0$, then X has normal structure (see [9]). Using Theorem 2.3, We can improve this result in the following form:

Corollary 2.4. *Let X be a Banach space with*

$$\rho_X(\tau) < \frac{\tau - 2 + \sqrt{\tau^2 + 4}}{2}$$

for some $\tau \in (0, 1]$. Then X has normal structure. In particular, if $A_2(X) < (1+\sqrt{5})/2$, then X and its dual X^* have normal structure.

In connection with a famous work of Jordan-von Neumann concerning inner products, the *Jordan-von Neumann constant* $C_{\mathrm{NJ}}(X)$ of X was introduced by Clarkson (cf. [1, 11]) as

$$C_{\mathrm{NJ}}(X) = \sup\left\{ \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ and not both zero} \right\}.$$

A relationship between $J(X)$ and $C_{\mathrm{NJ}}(X)$ is found in ([11] Theorem 3): $J(X) \leq \sqrt{2C_{\mathrm{NJ}}(X)}$.

In [5], Dhompongsa et al. proved the following inequality (2.3). We now restate this inequality without the ultra product technique and the fact $C_{\mathrm{NJ}}(X) = C_{\mathrm{NJ}}(X^*)$.

Theorem 2.5 ([5] Theorem 3.8). *Let X be a Banach space with the von Neumann-Jordan constant $C_{\mathrm{NJ}}(X)$. Then*

$$(WCS(X))^2 \geq \frac{2C_{\mathrm{NJ}}(X) + 1}{2(C_{\mathrm{NJ}}(X))^2}. \tag{2.3}$$

Proof. If $C_{\mathrm{NJ}}(X) = 2$, it suffices to note that $WCS(X) \geq 1$. Thus our estimates is a trivial one.

If $C_{\mathrm{NJ}}(X) < 2$, then X is reflexive. Let $\{x_n\}$ be a weakly null sequence in S_X. Assume that $d = \lim_{n,m,n\neq m} \|x_n - x_m\|$ exists and consider a normalized functional sequence $\{x_n^*\}$ such that $x_n^*(x_n) = 1$. Note that the reflexivity of X gurantees that there esists $x^* \in X^*$ such that $x_n^* \xrightarrow{w} x^*$. Let $\epsilon > 0$ and choose N large enough so that $|x^*(x_N)| < \epsilon/2$ and

$$d - \epsilon < \|x_N - x_m\| < d + \epsilon$$

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for all $m > N$. Note that

$$\lim_{n,m,n\neq m}\left\|\frac{x_n - x_m}{d+\epsilon}\right\| \le 1 \qquad \text{and} \qquad \left\|\frac{x_N}{d+\epsilon}\right\| \le 1.$$

Then by the definition of $R(1,X)$, we can choose $M > N$ large enough such that

$$\left\|\frac{x_N - x_M}{d+\epsilon}\right\| \le R(1,X)+\epsilon \le \sqrt{2C_{\mathrm{NJ}}(X)}+\epsilon, \quad |(x_M^* - x^*)(x_N)| < \epsilon/2,$$

and $|x_N^*(x_M)| < \epsilon$. Hence

$$|x_M^*(x_N)| < |(x_M^* - x^*)(x_N))| + |x^*(x_N)| < \epsilon.$$

Put $\alpha = \sqrt{2C_{\mathrm{NJ}}(X)}$, $x = \alpha^2(x_N - x_M)$, $y = x_N + x_M$. It follows that $\|x\| \le \alpha^2(d+\epsilon)$, $\|y\| \le (\alpha+\epsilon)(d+\epsilon)$, and also that

$$\begin{aligned}
\|x+y\| &= \|(\alpha^2+1)x_N - (\alpha^2-1)x_M\| \\
&\ge (\alpha^2+1)x_N^*(x_N) - (\alpha^2-1)x_N^*(x_M) \\
&\ge \alpha^2+1-3\epsilon, \\
\|x-y\| &= \|(\alpha^2+1)x_M - (\alpha^2-1)x_N\| \\
&\ge (\alpha^2+1)x_M^*(x_M) - (\alpha^2-1)x_M^*(x_N) \\
&\ge \alpha^2+1-3\epsilon.
\end{aligned}$$

Thus, from the definition of the von Neumann-Jordan constant,

$$\begin{aligned}
C_{\mathrm{NJ}}(X) &\ge \frac{2(\alpha^2+1-3\epsilon)^2}{2(\alpha^4(d+\epsilon)^2 + (\alpha+\epsilon)^2(d+\epsilon)^2)} \\
&= \frac{1}{(d+\epsilon)^2}\cdot\frac{(\alpha^2+1-3\epsilon)^2}{\alpha^4+(\alpha+\epsilon)^2}.
\end{aligned}$$

Since ϵ is arbitrary and $\alpha = \sqrt{2C_{\mathrm{NJ}}(X)}$, we get

$$C_{\mathrm{NJ}}(X) \geq \frac{1}{d^2}\left(1 + \frac{1}{\alpha^2}\right) = \frac{2C_{\mathrm{NJ}}(X) + 1}{d^2 \cdot 2C_{\mathrm{NJ}}(X)},$$

which implies that

$$d^2 \geq \frac{2C_{\mathrm{NJ}}(X) + 1}{2(C_{\mathrm{NJ}}(X))^2}.$$

Since the sequence $\{x_n\}$ is arbitrary, we obtain the inequality (2.3). □

Using Theorem 2.5, we can get a sufficient condition for X to have normal structure in terms of the von Neumann-Jordan constant.

Corollary 2.6 ([6, Theorem 3.16], [13, Theorem 2]). *Let X be a Banach space with $C_{\mathrm{NJ}}(X) < (1 + \sqrt{3})/2$. Then X and its dual X^* have normal structure.*

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