

SOME ESTIMATES ON THE WEAKLY CONVERGENT SEQUENCE COEFFICIENT IN BANACH SPACES

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Abstract

In this paper, we study the weakly convergent sequence coefficient and obtain its estimates for some parameters in Banach spaces, which give some sufficient conditions for a Banach space to have normal structure.

2000 Mathematics Subject Classification: 46B20.

Key words: Weakly convergent sequence coefficient; James constant; Von Neumann-Jordan constant; Modulus of smoothness.

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1. Introduction

A Banach space X said to have (weak) normal structure provided for every (weakly compact) closed bounded convex subset C of X with $\text{diam}(C) > 0$, contains a nondiametral point, i.e., there exists $x_0 \in C$ such that $\sup\{\|x - x_0\| : x \in C\} < \text{diam}(C)$. It is clear that normal structure and weak normal structure coincides when X is reflexive.

The weakly convergent sequence coefficient $WCS(X)$, a measure of weak normal structure, was introduced by Bynum in [3] as the following.

Definition 1.1. *The weakly convergent sequence coefficient of X is defined by*

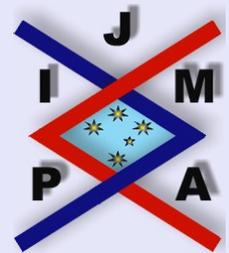
$$(1.1) \quad WCS(X)$$

$$= \inf \left\{ \frac{\text{diam}_a(\{x_n\})}{r_a(\{x_n\})} : \{x_n\} \text{ is a weakly convergent sequence} \right\},$$

where $\text{diam}_a(\{x_n\}) = \limsup_{k \rightarrow \infty} \{\|x_n - x_m\| : n, m \geq k\}$ is the asymptotic diameter of $\{x_n\}$ and $r_a(\{x_n\}) = \inf\{\limsup_{n \rightarrow \infty} \|x_n - y\| : y \in \bar{co}(\{x_n\})\}$ is the asymptotic radius of $\{x_n\}$.

One of the equivalent forms of $WCS(X)$ is

$$WCS(X) = \inf \left\{ \lim_{n, m, n \neq m} \|x_n - x_m\| : x_n \xrightarrow{w} 0, \|x_n\| = 1 \right. \\ \left. \text{and } \lim_{n, m, n \neq m} \|x_n - x_m\| \text{ exists} \right\}.$$



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Obviously, $1 \leq WCS(X) \leq 2$, and it is well known that $WCS(X) > 1$ implies that X has a weak normal structure.

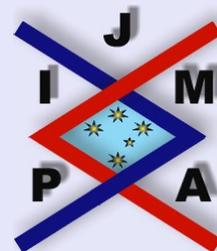
The constant $R(a, X)$, which is a generalized García-Falset coefficient [10], was introduced by Domínguez [7] as: For a given real number $a > 0$,

$$(1.2) \quad R(a, X) = \sup \left\{ \liminf_{n \rightarrow \infty} \|x + x_n\| \right\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq a$ and all weakly null sequences $\{x_n\} \subseteq B_X$ such that

$$(1.3) \quad \lim_{n, m, n \neq m} \|x_n - x_m\| \leq 1.$$

We shall assume throughout this paper that B_X and S_X to denote the unit ball and unit sphere of X , respectively. $x_n \xrightarrow{w} x$ stands for weak convergence of sequence $\{x_n\}$ in X to a point x in X .



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2. Main Results

The *James constant*, or the *nonsquare constant*, was introduced by Gao and Lau in [8] as

$$\begin{aligned} J(X) &= \sup\{\|x + y\| \wedge \|x - y\| : x, y \in S_X\} \\ &= \sup\{\|x + y\| \wedge \|x - y\| : x, y \in B_X\}. \end{aligned}$$

A relation between the constant $R(1, X)$ and the James constant $J(X)$ can be found in [6, 12]:

$$R(1, X) \leq J(X).$$

We now state an inequality between the James constant $J(X)$ and the weakly convergent sequence coefficient $WCS(X)$.

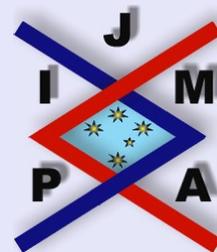
Theorem 2.1. *Let X be a Banach space with the James constant $J(X)$. Then*

$$(2.1) \quad WCS(X) \geq \frac{J(X) + 1}{(J(X))^2}.$$

Proof. If $J(X) = 2$, it suffices to note that $WCS(X) \geq 1$. Thus our estimate is a trivial one.

If $J(X) < 2$, then X is reflexive. Let $\{x_n\}$ be a weakly null sequence in S_X . Assume that $d = \lim_{n,m,n \neq m} \|x_n - x_m\|$ exists and consider a normalized functional sequence $\{x_n^*\}$ such that $x_n^*(x_n) = 1$. Note that the reflexivity of X guarantees, by passing through the subsequence, that there exists $x^* \in X^*$ such that $x_n^* \xrightarrow{w} x^*$. Let $0 < \epsilon < 1$ and choose N large enough so that $|x^*(x_N)| < \epsilon/2$ and

$$d - \epsilon < \|x_N - x_m\| < d + \epsilon$$



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for all $m > N$. Note that

$$\lim_{n,m,n \neq m} \left\| \frac{x_n - x_m}{d + \epsilon} \right\| \leq 1 \quad \text{and} \quad \left\| \frac{x_N}{d + \epsilon} \right\| \leq 1.$$

Then by the definition of $R(1, X)$, we can choose $M > N$ large enough such that

$$\left\| \frac{x_N + x_M}{d + \epsilon} \right\| \leq R(1, X) + \epsilon \leq J(X) + \epsilon, \quad |(x_M^* - x^*)(x_N)| < \epsilon/2,$$

and $|x_N^*(x_M)| < \epsilon$. Hence

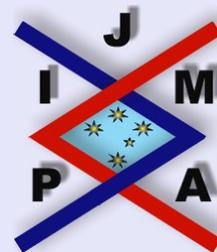
$$|x_M^*(x_N)| \leq |(x_M^* - x^*)(x_N)| + |x^*(x_N)| < \epsilon.$$

Put $\alpha = J(X)$,

$$x = \frac{x_N - x_M}{d + \epsilon}, \quad \text{and} \quad y = \frac{x_N + x_M}{(d + \epsilon)(\alpha + \epsilon)}.$$

It follows that $\|x\| \leq 1$, $\|y\| \leq 1$, and also that

$$\begin{aligned} \|x + y\| &= \frac{1}{(d + \epsilon)(\alpha + \epsilon)} \left\| (\alpha + 1 + \epsilon)x_N - (\alpha - 1 + \epsilon)x_M \right\| \\ &\geq \frac{1}{(d + \epsilon)(\alpha + \epsilon)} \left((\alpha + 1 + \epsilon)x_N^*(x_N) - (\alpha - 1 + \epsilon)x_N^*(x_M) \right) \\ &\geq \frac{\alpha + 1 - \epsilon}{(d + \epsilon)(\alpha + \epsilon)}, \end{aligned}$$



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$$\begin{aligned} \|x - y\| &= \frac{1}{(d + \epsilon)(\alpha + \epsilon)} \|(\alpha + 1 + \epsilon)x_M - (\alpha - 1 + \epsilon)x_N\| \\ &\geq \frac{1}{(d + \epsilon)(\alpha + \epsilon)} ((\alpha + 1 + \epsilon)x_M^*(x_M) - (\alpha - 1 + \epsilon)x_M^*(x_N)) \\ &\geq \frac{\alpha + 1 - \epsilon}{(d + \epsilon)(\alpha + \epsilon)}. \end{aligned}$$

Thus, from the definition of the James constant,

$$J(X) \geq \frac{\alpha + 1 - \epsilon}{(d + \epsilon)(\alpha + \epsilon)} = \frac{J(X) + 1 - \epsilon}{(d + \epsilon)(J(X) + \epsilon)}.$$

Letting $\epsilon \rightarrow 0$, we get

$$d \geq \frac{J(X) + 1}{(J(X))^2}.$$

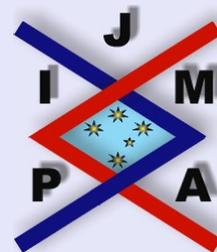
Since the sequence $\{x_n\}$ is arbitrary, we get the inequality (2.1). \square

As an application of Theorem 2.1, we can obtain a sufficient condition for X to have normal structure in terms of the James constant.

Corollary 2.2 ([4, Theorem 2.1]). *Let X be a Banach space with $J(X) < (1 + \sqrt{5})/2$. Then X has normal structure.*

The modulus of smoothness [14] of X is the function $\rho_X(\tau)$ defined by

$$\rho_X(\tau) = \sup \left\{ \frac{\|x + \tau y\| + \|x - \tau y\|}{2} - 1 : x, y \in S_X \right\}.$$



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It is readily seen that for any $x, y \in S_X$,

$$\|x \pm y\| \leq \|x \pm \tau y\| + (1 - \tau) \quad (0 < \tau \leq 1),$$

which implies that $J(X) \leq \rho_X(\tau) + 2 - \tau$.

In [2], Baronti et al. introduced a constant $A_2(X)$, which is defined by

$$A_2(X) = \rho_X(1) + 1 = \sup \left\{ \frac{\|x + y\| + \|x - y\|}{2} : x, y \in S_X \right\}.$$

It is worth noting that $A_2(X) = A_2(X^*)$.

We now state an inequality between the modulus of smoothness $\rho_X(\tau)$ and the weakly convergent sequence coefficient $WCS(X)$.

Theorem 2.3. *Let X be a Banach space with the modulus of smoothness $\rho_X(\tau)$. Then for any $0 < \tau \leq 1$,*

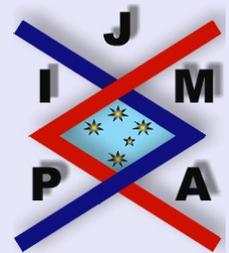
$$(2.2) \quad WCS(X) \geq \frac{\rho_X(\tau) + 2}{(\rho_X(\tau) + 1)(\rho_X(\tau) - \tau + 2)}.$$

Proof. Let $0 < \tau \leq 1$. If $\rho_X(\tau) = \tau$, it suffices to note that

$$\frac{\rho_X(\tau) + 2}{(\rho_X(\tau) + 1)(\rho_X(\tau) - \tau + 2)} = \frac{\tau + 2}{2(\tau + 1)} \leq 1.$$

Thus our estimate is a trivial one.

If $\rho_X(\tau) < \tau$, then X is reflexive. Let $\{x_n\}$ be a weakly null sequence in S_X . Assume that $d = \lim n, m, n \neq m \|x_n - x_m\|$ exists and consider a normalized



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functional sequence $\{x_n^*\}$ such that $x_n^*(x_n) = 1$. Note that the reflexivity of X guarantees that there exists $x^* \in X^*$ such that $x_n^* \xrightarrow{w} x^*$. Let $\epsilon > 0$ and x_M, x_N, x and y selected as in Theorem 2.1. Similarly, we get

$$\|x \pm \tau y\| \geq \frac{\alpha(\tau) + \tau - \epsilon}{(d + \epsilon)(\alpha(\tau) + \epsilon)},$$

where $\alpha(\tau) = \rho_X(\tau) + 2 - \tau$. Then by the definition of $\rho_X(\tau)$, we obtain

$$\rho_X(\tau) \geq \frac{\alpha(\tau) + \tau - \epsilon}{(d + \epsilon)(\alpha(\tau) + \epsilon)} - 1.$$

Letting $\epsilon \rightarrow 0$,

$$\rho_X(\tau) + 1 \geq \frac{\alpha(\tau) + \tau}{d\alpha(\tau)} = \frac{\rho_X(\tau) + 2}{d(\rho_X(\tau) - \tau + 2)},$$

which gives that

$$d \geq \frac{\rho_X(\tau) + 2}{(\rho_X(\tau) + 1)(\rho_X(\tau) - \tau + 2)}.$$

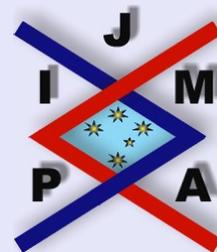
Since the sequence $\{x_n\}$ is arbitrary, we get the inequality (2.2). \square

It is known that if $\rho_X(\tau) < \tau/2$ for some $\tau > 0$, then X has normal structure (see [9]). Using Theorem 2.3, We can improve this result in the following form:

Corollary 2.4. *Let X be a Banach space with*

$$\rho_X(\tau) < \frac{\tau - 2 + \sqrt{\tau^2 + 4}}{2}$$

for some $\tau \in (0, 1]$. Then X has normal structure. In particular, if $A_2(X) < (1 + \sqrt{5})/2$, then X and its dual X^* have normal structure.



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In connection with a famous work of Jordan-von Neumann concerning inner products, the *Jordan-von Neumann constant* $C_{\text{NJ}}(X)$ of X was introduced by Clarkson (cf. [1, 11]) as

$$C_{\text{NJ}}(X) = \sup \left\{ \frac{\|x + y\|^2 + \|x - y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X \text{ and not both zero} \right\}.$$

A relationship between $J(X)$ and $C_{\text{NJ}}(X)$ is found in ([11] Theorem 3): $J(X) \leq \sqrt{2C_{\text{NJ}}(X)}$.

In [5], Dhompongsa et al. proved the following inequality (2.3). We now restate this inequality without the ultra product technique and the fact $C_{\text{NJ}}(X) = C_{\text{NJ}}(X^*)$.

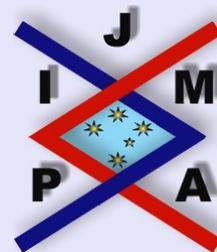
Theorem 2.5 ([5] Theorem 3.8). *Let X be a Banach space with the von Neumann-Jordan constant $C_{\text{NJ}}(X)$. Then*

$$(2.3) \quad (WCS(X))^2 \geq \frac{2C_{\text{NJ}}(X) + 1}{2(C_{\text{NJ}}(X))^2}.$$

Proof. If $C_{\text{NJ}}(X) = 2$, it suffices to note that $WCS(X) \geq 1$. Thus our estimates is a trivial one.

If $C_{\text{NJ}}(X) < 2$, then X is reflexive. Let $\{x_n\}$ be a weakly null sequence in S_X . Assume that $d = \lim_{n,m,n \neq m} \|x_n - x_m\|$ exists and consider a normalized functional sequence $\{x_n^*\}$ such that $x_n^*(x_n) = 1$. Note that the reflexivity of X guarantees that there exists $x^* \in X^*$ such that $x_n^* \xrightarrow{w} x^*$. Let $\epsilon > 0$ and choose N large enough so that $|x^*(x_N)| < \epsilon/2$ and

$$d - \epsilon < \|x_N - x_m\| < d + \epsilon$$



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for all $m > N$. Note that

$$\lim_{n,m,n \neq m} \left\| \frac{x_n - x_m}{d + \epsilon} \right\| \leq 1 \quad \text{and} \quad \left\| \frac{x_N}{d + \epsilon} \right\| \leq 1.$$

Then by the definition of $R(1, X)$, we can choose $M > N$ large enough such that

$$\left\| \frac{x_N - x_M}{d + \epsilon} \right\| \leq R(1, X) + \epsilon \leq \sqrt{2C_{NJ}(X)} + \epsilon, \quad |(x_M^* - x^*)(x_N)| < \epsilon/2,$$

and $|x_N^*(x_M)| < \epsilon$. Hence

$$|x_M^*(x_N)| < |(x_M^* - x^*)(x_N)| + |x^*(x_N)| < \epsilon.$$

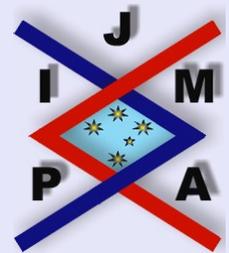
Put $\alpha = \sqrt{2C_{NJ}(X)}$, $x = \alpha^2(x_N - x_M)$, $y = x_N + x_M$. It follows that $\|x\| \leq \alpha^2(d + \epsilon)$, $\|y\| \leq (\alpha + \epsilon)(d + \epsilon)$, and also that

$$\begin{aligned} \|x + y\| &= \|(\alpha^2 + 1)x_N - (\alpha^2 - 1)x_M\| \\ &\geq (\alpha^2 + 1)x_N^*(x_N) - (\alpha^2 - 1)x_M^*(x_M) \\ &\geq \alpha^2 + 1 - 3\epsilon, \end{aligned}$$

$$\begin{aligned} \|x - y\| &= \|(\alpha^2 + 1)x_M - (\alpha^2 - 1)x_N\| \\ &\geq (\alpha^2 + 1)x_M^*(x_M) - (\alpha^2 - 1)x_N^*(x_N) \\ &\geq \alpha^2 + 1 - 3\epsilon. \end{aligned}$$

Thus, from the definition of the von Neumann-Jordan constant,

$$\begin{aligned} C_{NJ}(X) &\geq \frac{2(\alpha^2 + 1 - 3\epsilon)^2}{2(\alpha^4(d + \epsilon)^2 + (\alpha + \epsilon)^2(d + \epsilon)^2)} \\ &= \frac{1}{(d + \epsilon)^2} \cdot \frac{(\alpha^2 + 1 - 3\epsilon)^2}{\alpha^4 + (\alpha + \epsilon)^2}. \end{aligned}$$



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Since ϵ is arbitrary and $\alpha = \sqrt{2C_{\text{NJ}}(X)}$, we get

$$C_{\text{NJ}}(X) \geq \frac{1}{d^2} \left(1 + \frac{1}{\alpha^2}\right) = \frac{2C_{\text{NJ}}(X) + 1}{d^2 \cdot 2C_{\text{NJ}}(X)},$$

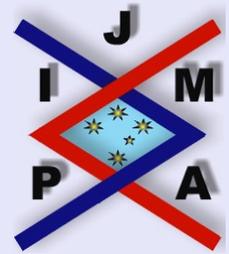
which implies that

$$d^2 \geq \frac{2C_{\text{NJ}}(X) + 1}{2(C_{\text{NJ}}(X))^2}.$$

Since the sequence $\{x_n\}$ is arbitrary, we obtain the inequality (2.3). \square

Using Theorem 2.5, we can get a sufficient condition for X to have normal structure in terms of the von Neumann-Jordan constant.

Corollary 2.6 ([6, Theorem 3.16], [13, Theorem 2]). *Let X be a Banach space with $C_{\text{NJ}}(X) < (1 + \sqrt{3})/2$. Then X and its dual X^* have normal structure.*



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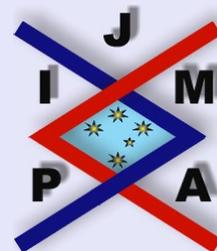
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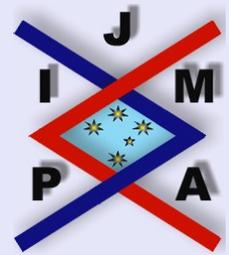
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