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A NOTE ON ABSOLUTE NÖRLUND SUMMABILITY

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ABSTRACT. In this paper a main theorem on $|N, p_n|_k$ summability factors, which generalizes a result of Bor [2] on $|N, p_n|$ summability factors, has been proved.

Key words and phrases: Nörlund summability, summability factors, power increasing sequences.

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1. INTRODUCTION

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). A positive sequence (γ_n) is said to be a quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$Kn^\beta \gamma_n \geq m^\beta \gamma_m \tag{1.1}$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is a quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. We denote by $\mathcal{BV}_\mathcal{O}$ the $\mathcal{BV} \cap \mathcal{C}_\mathcal{O}$, where $\mathcal{C}_\mathcal{O}$ and $\mathcal{BV}$ are the null sequences and sequences with bounded variation, respectively.

Let $\sum a_n$ be a given infinite series with the sequence of partial sums (s_n) and $w_n = na_n$. By u_n^α and t_n^α we denote the n-th Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (w_n), respectively.

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$, if (see [4])

$$\sum_{n=1}^{\infty} n^{k-1} \left| u_n^\alpha - u_{n-1}^\alpha \right|^k = \sum_{n=1}^{\infty} \frac{1}{n} \left| t_n^\alpha \right|^k < \infty. \tag{1.2}$$

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Let (p_n) be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n \neq 0, \ (n \geq 0). \tag{1.3}$$

The sequence-to-sequence transformation

$$\sigma_n = \frac{1}{P_n} \sum_{v=0}^{n} p_{n-v} s_v \tag{1.4}$$

defines the sequence (σ_n) of the Nörlund mean of the sequence (s_n), generated by the sequence of coefficients (p_n). The series $\sum a_n$ is said to be summable $|N, p_n|_k$, $k \geq 1$, if (see [3])

$$\sum_{n=1}^{\infty} n^{k-1} \left| \sigma_n - \sigma_{n-1} \right|^k < \infty. \tag{1.5}$$

In the special case when

$$p_n = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\Gamma(n+1)}, \ \alpha \geq 0 \tag{1.6}$$

the Nörlund mean reduces to the (C, α) mean and $|N, p_n|_k$ summability becomes $|C, \alpha|_k$ summability. For $p_n = 1$ and $P_n = n$, we get the $(C, 1)$ mean and then $|N, p_n|_k$ summability becomes $|C, 1|_k$ summability. For any sequence (λ_n), we write $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

The known results. Concerning the $|C, 1|_k$ and $|N, p_n|_k$ summabilities Varma [6] has proved the following theorem.

Theorem A. *Let $p_0 > 0$, $p_n \geq 0$ and (p_n) be a non-increasing sequence. If $\sum a_n$ is summable $|C, 1|_k$, then the series $\sum a_n P_n (n+1)^{-1}$ is summable $|N, p_n|_k$, $k \geq 1$.*

Quite recently Bor [2] has proved the following theorem.

Theorem B. *Let (p_n) be as in Theorem A, and let (X_n) be a quasi β-power increasing sequence with some $0 < \beta < 1$. If*

$$\sum_{v=1}^{n} \frac{1}{v} |t_v| = O(X_n) \quad \textit{as } n \to \infty, \tag{1.7}$$

and the sequences (λ_n) and (β_n) satisfy the following conditions

$$X_n \lambda_n = O(1), \tag{1.8}$$

$$|\Delta\lambda_n| \leq \beta_n, \tag{1.9}$$

$$\beta_n \to 0, \tag{1.10}$$

$$\sum n X_n |\Delta\beta_n| < \infty, \tag{1.11}$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N, p_n|$.

2. MAIN RESULT

The aim of this paper is to generalize Theorem B for $|N,p_n|_k$ summability. Now we shall prove the following theorem.

Theorem 2.1. *Let (p_n) be as in Theorem A, and let (X_n) be a quasi β-power increasing sequence with some $0<\beta<1$. If*

$$\sum_{v=1}^{n} \frac{1}{v} |t_v|^k = O(X_n) \quad \text{as } n \to \infty, \tag{2.1}$$

and the sequences (λ_n) and (β_n) satisfy the conditions from (1.8) to (1.11) of Theorem B; further suppose that

$$(\lambda_n) \in \mathcal{BV}_{\mathcal{O}}, \tag{2.2}$$

then the series $\sum a_n P_n \lambda_n (n+1)^{-1}$ is summable $|N,p_n|_k$, $k \geq 1$.

Remark 2.2. It should be noted that if we take $k=1$, then we get Theorem B. In this case condition (2.2) is not needed.

We need the following lemma for the proof of our theorem.

Lemma 2.3 ([5]). *Except for the condition (2.2), under the conditions on (X_n), (λ_n) and (β_n) as taken in the statement of the theorem, the following conditions hold when (1.11) is satisfied:*

$$n\beta_n X_n = O(1) \; as \; n \to \infty, \tag{2.3}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{2.4}$$

3. PROOF OF THEOREM 2.1

In order to prove the theorem, we need consider only the special case in which (N,p_n) is $(C,1)$, that is, we shall prove that $\sum a_n\lambda_n$ is summable $|C,1|_k$. Our theorem will then follow by means of Theorem A. Let T_n be the $n-$th $(C,1)$ mean of the sequence $(na_n\lambda_n)$, that is,

$$T_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v \lambda_v. \tag{3.1}$$

Using Abel's transformation, we have

$$\begin{aligned} T_n &= \frac{1}{n+1} \sum_{v=1}^{n} v a_v \lambda_v = \frac{1}{n+1} \sum_{v=1}^{n-1} \Delta\lambda_v (v+1) t_v + \lambda_n t_n \\ &= T_{n,1} + T_{n,2}, \text{ say.} \end{aligned}$$

To complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{1}{n} |T_{n,r}|^k < \infty \quad \text{for } r = 1,2, \text{ by (1.2).} \tag{3.2}$$

Now, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1} \frac{1}{n} |T_{n,1}|^k &\le \sum_{n=2}^{m+1} \frac{1}{n(n+1)^k} \left\{ \sum_{v=1}^{n-1} \frac{v+1}{v} v |\Delta\lambda_v| |t_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^{k+1}} \left\{ \sum_{v=1}^{n-1} v |\Delta\lambda_v| |t_v| \right\}^k \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \left\{ \sum_{v=1}^{n-1} v |\Delta\lambda_v| |t_v|^k \right\} \times \left\{ \frac{1}{n} \sum_{v=1}^{n-1} v |\Delta\lambda_v| \right\}^{k-1} \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \sum_{v=1}^{n-1} v |\Delta\lambda_v| |t_v|^k \qquad \text{(by (2.2))} \\
&= O(1) \sum_{n=2}^{m+1} \frac{1}{n^2} \left\{ \sum_{v=1}^{n-1} v\beta_v |t_v|^k \right\} \qquad \text{(by (1.9))} \\
&= O(1) \sum_{v=1}^{m} v\beta_v |t_v|^k \sum_{n=v+1}^{m+1} \frac{1}{n^2} = O(1) \sum_{v=1}^{m} v\beta_v \frac{|t_v|^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} \Delta(v\beta_v) \sum_{r=1}^{v} \frac{|t_r|^k}{r} + O(1) m\beta_m \sum_{v=1}^{m} \frac{|t_v|^k}{v} \\
&= O(1) \sum_{v=1}^{m-1} |\Delta(v\beta_v)| X_v + O(1) m\beta_m X_m \qquad \text{(by (2.1))} \\
&= O(1) \sum_{v=1}^{m-1} |(v+1)\Delta\beta_v - \beta_v| X_v + O(1) m\beta_m X_m \\
&= O(1) \sum_{v=1}^{m-1} v |\Delta\beta_v| X_v + O(1) \sum_{v=1}^{m-1} |\beta_v| X_v + O(1) m\beta_m X_m \\
&= O(1) \quad \text{as } m \to \infty,
\end{aligned}$$

in view of (1.11), (2.3) and (2.4).

Again

$$\begin{aligned}
\sum_{n=1}^{m} \frac{1}{n} |T_{n,2}|^k &= \sum_{n=1}^{m} |\lambda_n|^k \frac{|t_n|^k}{n} \\
&= \sum_{n=1}^{m} |\lambda_n|^{k-1} |\lambda_n| \frac{|t_n|^k}{n} = O(1) \sum_{n=1}^{m} |\lambda_n| \frac{|t_n|^k}{n} \qquad \text{(by (2.2))} \\
&= O(1) \sum_{n=1}^{m-1} \Delta |\lambda_n| \sum_{v=1}^{n} \frac{|t_v|^k}{v} + O(1) |\lambda_m| \sum_{n=1}^{m} \frac{|t_n|^k}{n} \\
&= O(1) \sum_{n=1}^{m-1} |\Delta\lambda_n| X_n + O(1) |\lambda_m| X_m \qquad \text{(by (2.1))} \\
&= O(1) \sum_{n=1}^{m-1} \beta_n X_n + O(1) |\lambda_m| X_m = O(1) \quad \text{as } m \to \infty,
\end{aligned}$$

by virtue of (1.8) and (2.4). This completes the proof of the theorem. □

REFERENCES

[1] S. ALJANČIĆ AND D. ARANDELOVIĆ, *O*-regularly varying functions, *Publ. Inst. Math.*, **22** (1977), 5–22.

[2] H. BOR, Absolute Nörlund summability factors, *J. Inequal. Pure Appl. Math.*, **6**(3) (2005), Art. 62. [ONLINE `http://jipam.vu.edu.au/article.php?sid=535`].

[3] D. BORWEIN AND F.P. CASS, Strong Nörlund summability, *Math. Zeith.*, **103** (1968), 94–111.

[4] T.M. FLETT, On an extension of absolute summability and some theorems of Littlewood and Paley, *Proc. London Math. Soc.*, **7** (1957), 113–141.

[5] L. LEINDLER, A new application of quasi power increasing sequences, *Publ. Math. Debrecen*, **58** (2001), 791–796.

[6] R.S. VARMA, On the absolute Nörlund summability factors, *Riv. Math. Univ. Parma* (4), **3** (1977), 27–33.