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ON THE DEGREE OF STRONG APPROXIMATION OF INTEGRABLE FUNCTIONS

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ABSTRACT. We show the strong approximation version of some results of L. Leindler [3] connected with the theorems of P. Chandra [1, 2].

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1. INTRODUCTION

Let L^p $(1 < p < \infty)$ $[C]$ be the class of all 2π–periodic real–valued functions integrable in the Lebesgue sense with p–th power [continuous] over $Q = [-\pi, \pi]$ and let $X^p = L^p$ when $1 < p < \infty$ or $X^p = C$ when $p = \infty$. Let us define the norm of $f \in X^p$ as

$$\|f\|_{X^p} = \|f(\cdot)\|_{X^p} = \begin{cases} \left(\int_Q |f(x)|^p \, dx\right)^{\frac{1}{p}} & \text{when } 1 < p < \infty, \\ \sup_{x \in Q} |f(x)| & \text{when } p = \infty, \end{cases}$$

and consider its trigonometric Fourier series

$$Sf(x) = \frac{a_o(f)}{2} + \sum_{\nu=0}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)$$

with the partial sums $S_k f$.

Let $A := (a_{n,k})$ $(k, n = 0, 1, 2, \dots)$ be a lower triangular infinite matrix of real numbers and let the $A-$transforms of $(S_k f)$ be given by

$$T_{n,A} f(x) := \left| \sum_{k=0}^{n} a_{n,k} S_k f(x) - f(x) \right| \qquad (n = 0, 1, 2, \dots)$$

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and

$$H_{n,A}^{q}f\left(x\right):=\left\{\sum_{k=0}^{n}a_{n,k}\left|S_{k}f\left(x\right)-f\left(x\right)\right|^{q}\right\}^{\frac{1}{q}}\qquad\left(q>0,\ n=0,1,2,\dots\right).$$

As a measure of approximation, by the above quantities we use the pointwise characteristic

$$w_{x}f(\delta)_{L^{p}}:=\left\{\frac{1}{\delta}\int_{0}^{\delta}\left|\varphi_{x}\left(t\right)\right|^{p}dt\right\}^{\frac{1}{p}},$$

where

$$\varphi_{x}\left(t\right):=f\left(x+t\right)+f\left(x-t\right)-2f\left(x\right).$$

$w_{x}f(\delta)_{L^{p}}$ is constructed based on the definition of Lebesgue points ($L^{p}-$points), and the modulus of continuity for f in the space X^{p} defined by the formula

$$\omega f\left(\delta\right)_{X^{p}}:=\sup_{0\leq|h|\leq\delta}\left\|\varphi\cdot\left(h\right)\right\|_{X^{p}}.$$

We can observe that with $\widetilde{p}\geq p$, for $f\in X^{\widetilde{p}}$, by the Minkowski inequality

$$\left\|w\cdot f(\delta)_{p}\right\|_{X^{\widetilde{p}}}\leq\omega f\left(\delta\right)_{X^{\widetilde{p}}}.$$

The deviation $T_{n,A}f$ was estimated by P. Chandra [1, 2] in the norm of $f\in C$ and for monotonic sequences $a_{n}=\left(a_{n,k}\right)$. These results were generalized by L. Leindler [3] who considered the sequences of bounded variation instead of monotonic ones. In this note we shall consider the strong means $H_{n,A}^{q}f$ and the integrable functions. We shall also give some results on norm approximation.

By K we shall designate either an absolute constant or a constant depending on some parameters, not necessarily the same of each occurrence.

2. Statement of the Results

Let us consider a function w_{x} of modulus of continuity type on the interval $[0,+\infty)$, i.e., a nondecreasing continuous function having the following properties: $w_{x}\left(0\right)=0,\ w_{x}\left(\delta_{1}+\delta_{2}\right)\leq w_{x}\left(\delta_{1}\right)+w_{x}\left(\delta_{2}\right)$ for any $0\leq\delta_{1}\leq\delta_{2}\leq\delta_{1}+\delta_{2}$ and let

$$L^{p}\left(w_{x}\right)=\left\{f\in L^{p}:w_{x}f\left(\delta\right)_{L^{p}}\leq w_{x}\left(\delta\right)\right\}.$$

We can now formulate our main results.

To start with, we formulate the results on pointwise approximation.

Theorem 2.1. *Let $a_{n}=\left(a_{n,m}\right)$ satisfy the following conditions:*

$$a_{n,m}\geq0,\qquad\sum_{k=0}^{n}a_{n,k}=1\tag{2.1}$$

and

$$\sum_{k=0}^{m-1}\left|a_{n,k}-a_{n,k+1}\right|\leq Ka_{n,m},\tag{2.2}$$

where

$$m=0,1,\dots,n\quad\textit{and}\quad n=0,1,2,\dots\qquad\left(\sum\nolimits_{k=0}^{-1}=0\right).$$

Suppose w_{x} is such that

$$\left\{u^{\frac{p}{q}}\int_{u}^{\pi}\frac{\left(w_{x}(t)\right)^{p}}{t^{1+\frac{p}{q}}}dt\right\}^{\frac{1}{p}}=O\left(uH_{x}\left(u\right)\right)\quad\textit{as}\quad u\rightarrow0+,\tag{2.3}$$

where $H_x(u) \geq 0$, $1 < p \leq q$ *and*

$$\int_0^t H_x(u)\,du = O(tH_x(t)) \quad \text{as} \quad t \to 0+. \tag{2.4}$$

If $f \in L^p(w_x)$, *then*

$$H_{n,A}^{q'} f(x) = O(a_{n,n} H_x(a_{n,n}))$$

with $q' \in (0, q]$ *and* q *such that* $1 < q(q-1) \leq p \leq q$.

Theorem 2.2. *Let* (2.1), (2.2) *and* (2.3) *hold. If* $f \in L^p(w_x)$ *then*

$$H_{n,A}^{q'} f(x) = O\left(w_x\left(\frac{\pi}{n+1}\right)\right) + O\left(a_{n,n} H_x\left(\frac{\pi}{n+1}\right)\right)$$

and if, in addition, (2.4) *holds then*

$$H_{n,A}^{q'} f(x) = O\left(a_{n,n} H_x\left(\frac{\pi}{n+1}\right)\right)$$

with $q' \in (0, q]$ *and* q *such that* $1 < q(q-1) \leq p \leq q$.

Theorem 2.3. *Let* (2.1), (2.3), (2.4) *and*

$$\sum_{k=m}^{\infty} |a_{n,k} - a_{n,k+1}| \leq K a_{n,m}, \tag{2.5}$$

where

$$m = 0, 1, \ldots, n \ \text{ and } \ n = 0, 1, 2, \ldots$$

hold. If $f \in L^p(w_x)$ *then*

$$H_{n,A}^{q'} f(x) = O(a_{n,0} H_x(a_{n,0}))$$

with $q' \in (0, q]$ *and* q *such that* $1 < q(q-1) \leq p \leq q$.

Theorem 2.4. *Let us assume that* (2.1), (2.3) *and* (2.5) *hold. If* $f \in L^p(w_x)$, *then*

$$H_{n,A}^{q'} f(x) = O\left(w_x\left(\frac{\pi}{n+1}\right)\right) + O\left(a_{n,0} H_x\left(\frac{\pi}{n+1}\right)\right).$$

If, in addition, (2.4) *holds then*

$$H_{n,A}^{q'} f(x) = O\left(a_{n,0} H_x\left(\frac{\pi}{n+1}\right)\right)$$

with $q' \in (0, q]$ *and* q *such that* $1 < q(q-1) \leq p \leq q$.

Consequently, we formulate the results on norm approximation.

Theorem 2.5. *Let* $a_n = (a_{n,m})$ *satisfy the conditions* (2.1) *and* (2.2). *Suppose* $\omega f(\cdot)_{X^{\widetilde{p}}}$ *is such that*

$$\left\{u^{\frac{p}{q}} \int_u^{\pi} \frac{(\omega f(t)_{X^{\widetilde{p}}})^p}{t^{1+\frac{p}{q}}} dt\right\}^{\frac{1}{p}} = O(uH(u)) \quad \text{as} \quad u \to 0+ \tag{2.6}$$

holds, with $1 < p \leq q$ *and* $\widetilde{p} \geq p$, *where* H (≥ 0) *instead of* H_x *satisfies the condition* (2.4). *If* $f \in X^{\widetilde{p}}$ *then*

$$\left\| H_{n,A}^{q'} f(\cdot) \right\|_{X^{\widetilde{p}}} = O(a_{n,n} H(a_{n,n}))$$

with $q' \leq q$ *and* $p \leq \widetilde{p}$ *such that* $1 < q(q-1) \leq p \leq q$.

Theorem 2.6. *Let* (2.1), (2.2) *and* (2.6) *hold. If* $f \in X^{\widetilde{p}}$ *then*

$$\left\| H_{n,A}^{q'} f(\cdot) \right\|_{X^{\widetilde{p}}} = O\left(\omega f \left(\frac{\pi}{n+1} \right)_{X^{\widetilde{p}}} \right) + O\left(a_{n,n} H \left(\frac{\pi}{n+1} \right) \right).$$

If, in addition, $H\ (\geq 0)$ *instead of* H_x *satisfies the condition* (2.4) *then*

$$\left\| H_{n,A}^{q'} f(\cdot) \right\|_{X^{\widetilde{p}}} = O\left(a_{n,n} H \left(\frac{\pi}{n+1} \right) \right)$$

with $q' \leq q$ *and* $p \leq \widetilde{p}$ *such that* $1 < q(q-1) \leq p \leq q$.

Theorem 2.7. *Let* (2.1), (2.4) *with a function* $H\ (\geq 0)$ *instead of* H_x, (2.5) *and* (2.6) *hold. If* $f \in X^{\widetilde{p}}$ *then*

$$\left\| H_{n,A}^{q'} f(\cdot) \right\|_{X^{\widetilde{p}}} = O\left(a_{n,0} H\left(a_{n,0} \right) \right)$$

with $q' \leq q$ *and* $p \leq \widetilde{p}$ *such that* $1 < q(q-1) \leq p \leq q$.

Theorem 2.8. *Let* (2.1), (2.5) *and* (2.6) *hold. If* $f \in X^{\widetilde{p}}$ *then*

$$\left\| H_{n,A}^{q'} f(\cdot) \right\|_{X^{\widetilde{p}}} = O\left(\omega f \left(\frac{\pi}{n+1} \right)_{X^{\widetilde{p}}} \right) + O\left(a_{n,0} H \left(\frac{\pi}{n+1} \right) \right).$$

If, in addition, $H\ (\geq 0)$ *instead of* H_x *satisfies the condition* (2.4), *then*

$$\left\| H_{n,A}^{q'} f(\cdot) \right\|_{X^{\widetilde{p}}} = O\left(a_{n,0} H \left(\frac{\pi}{n+1} \right) \right)$$

with $q' \leq q$ *and* $p \leq \widetilde{p}$ *such that* $1 < q(q-1) \leq p \leq q$.

3. AUXILIARY RESULTS

In tis section we denote by ω a function of modulus of continuity type.

Lemma 3.1. *If* (2.3) *with* $0 < p \leq q$ *and* (2.4) *with functions* ω *and* $H\ (\geq 0)$ *instead of* w_x *and* H_x, *respectively, hold then*

(3.1) $$\int_0^u \frac{\omega(t)}{t} dt = O(uH(u)) \qquad (u \to 0+).$$

Proof. Integrating by parts in the above integral we obtain

$$\begin{aligned}
\int_0^u \frac{\omega(t)}{t} dt &= \int_0^u t \frac{d}{dt} \left(\int_t^\pi \frac{\omega(s)}{s^2} ds \right) dt \\
&= \left[-t \int_t^\pi \frac{\omega(s)}{s^2} ds \right]_0^u + \int_0^u \left(\int_t^\pi \frac{\omega(s)}{s^2} ds \right) dt \\
&\leq u \int_u^\pi \frac{\omega(s)}{s^2} ds + \int_0^u \left(\int_t^\pi \frac{\omega(s)}{s^2} ds \right) dt \\
&= u \int_u^\pi \frac{\omega(s)}{s^{1+\frac{p}{q}+1-\frac{p}{q}}} ds + \int_0^u \left(\int_t^\pi \frac{\omega(s)}{s^{1+\frac{p}{q}+1-\frac{p}{q}}} ds \right) dt \\
&\leq u^{\frac{p}{q}} \int_u^\pi \frac{\omega(s)}{s^{1+p/q}} ds + \int_0^u \frac{1}{t} \left(t^{\frac{p}{q}} \int_t^\pi \frac{(\omega(s))^p}{s^{1+p/q}} ds \right)^{\frac{1}{p}} dt,
\end{aligned}$$

since $1 - \frac{p}{q} \geq 0$. Using our assumptions we have

$$\int_0^u \frac{\omega(t)}{t} dt = O(uH(u)) + \int_0^u \frac{1}{t} O(tH(t))\, dt = O(uH(u))$$

and thus the proof is completed. □

Lemma 3.2 ([4, Theorem 5.20 II, Ch. XII]). *Suppose that* $1 < q(q-1) \leq p \leq q$ *and* $\xi = 1/p + 1/q - 1$. *If* $\left|t^{-\xi} g(t)\right| \in L^p$ *then*

$$\left\{ \frac{|a_o(g)|^q}{2} + \sum_{k=0}^{\infty} (|a_k(g)|^q + |b_k(g)|^q) \right\}^{\frac{1}{q}} \leq K \left\{ \int_{-\pi}^{\pi} \left|t^{-\xi} g(t)\right|^p dt \right\}^{\frac{1}{p}}. \tag{3.2}$$

4. PROOFS OF THE RESULTS

Since $H^q_{n,A} f$ is the monotonic function of q we shall consider, in all our proofs, the quantity $H^q_{n,A} f$ instead of $H^{q'}_{n,A} f$.

Proof of Theorem 2.1. Let

$$\begin{aligned} H^q_{n,A} f(x) &= \left\{ \sum_{k=0}^{n} a_{n,k} \left| \frac{1}{\pi} \int_0^{\pi} \varphi_x(t) \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q \right\}^{\frac{1}{q}} \\ &\leq \left\{ \sum_{k=0}^{n} a_{n,k} \left| \frac{1}{\pi} \int_0^{a_{n,n}} \varphi_x(t) \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q \right\}^{\frac{1}{q}} \\ &\qquad + \left\{ \sum_{k=0}^{n} a_{n,k} \left| \frac{1}{\pi} \int_{a_{n,n}}^{\pi} \varphi_x(t) \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q \right\}^{\frac{1}{q}} \\ &= I(a_{n,n}) + J(a_{n,n}) \end{aligned}$$

and, by (2.1) , integrating by parts, we obtain,

$$\begin{aligned} I(a_{n,n}) &\leq \int_0^{a_{n,n}} \frac{|\varphi_x(t)|}{2t} dt \\ &= \int_0^{a_{n,n}} \frac{1}{2t} \frac{d}{dt} \left(\int_0^t |\varphi_x(s)|\, ds \right) dt \\ &= \frac{1}{2a_{n,n}} \int_0^{a_{n,n}} |\varphi_x(t)|\, dt + \int_0^{a_{n,n}} \frac{w_x f(t)_1}{2t} dt \\ &= \frac{1}{2} \left(w_x f(a_{n,n})_1 + \int_0^{a_{n,n}} \frac{w_x f(t)_1}{t} dt \right) \\ &= K \left(a_{n,n} \int_{a_{n,n}}^{\pi} \frac{w_x f(t)_1}{t^2} dt + \int_0^{a_{n,n}} \frac{w_x f(t)_1}{t} dt \right) \\ &\leq K \left(a_{n,n} \int_{a_{n,n}}^{\pi} \frac{(w_x f(t)_{L^1})^p}{t^2} dt \right)^{\frac{1}{p}} + K \left(\int_0^{a_{n,n}} \frac{w_x f(t)_{L^1}}{t} dt \right) \\ &\leq K \left(a_{n,n}^{p/q} \int_{a_{n,n}}^{\pi} \frac{(w_x f(t)_{L^p})^p}{t^{1+p/q}} dt \right)^{\frac{1}{p}} + K \left(\int_0^{a_{n,n}} \frac{w_x f(t)_{L^p}}{t} dt \right). \end{aligned}$$

Since $f \in L^p(w_x)$ and (2.4) holds, Lemma 3.1 and (2.3) give

$$I(a_{n,n}) = O(a_{n,n} H_x(a_{n,n})).$$

The Abel transformation shows that

$$(J(a_{n,n}))^q = \sum_{k=0}^{n-1} (a_{n,k} - a_{n,k+1}) \sum_{\nu=0}^{k} \left| \frac{1}{\pi} \int_{a_{n,n}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q \\ + a_{n,n} \sum_{\nu=0}^{n} \left| \frac{1}{\pi} \int_{a_{n,n}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q,$$

whence, by (2.2),

$$(J(a_{n,n}))^q \leq (K+1)\, a_{n,n} \sum_{\nu=0}^{n} \left| \frac{1}{\pi} \int_{a_{n,n}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q.$$

Using inequality (3.2), we obtain

$$J(a_{n,n}) \leq K (a_{n,n})^{\frac{1}{q}} \left\{ \int_{a_{n,n}}^{\pi} \frac{|\varphi_x(t)|^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}.$$

Integrating by parts, we have

$$J(a_{n,n}) \leq K (a_{n,n})^{\frac{1}{q}} \left\{ \left[\frac{1}{t^{p/q}} \left(w_x f(t)_{L^p}\right)^p \right]_{t=a_{n,n}}^{\pi} \right. \\ \left. + \left(1+\frac{p}{q}\right) \int_{a_{n,n}}^{\pi} \frac{\left(w_x f(t)_{L^p}\right)^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}} \\ \leq K (a_{n,n})^{\frac{1}{q}} \left\{ \left(w_x f(\pi)_{L^p}\right) + \int_{a_{n,n}}^{\pi} \frac{\left(w_x f(t)_{L^p}\right)^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}.$$

Since $f \in L^p(w_x)$, by (2.3),

$$J(a_{n,n}) \leq K (a_{n,n})^{\frac{1}{q}} \left\{ (w_x(\pi)) + \int_{a_{n,n}}^{\pi} \frac{(w_x(t))^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}} \\ \leq K \left\{ (a_{n,n})^{\frac{p}{q}} \int_{a_{n,n}}^{\pi} \frac{(w_x(t))^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}} \\ = O\left(a_{n,n} H_x(a_{n,n})\right).$$

Thus our result is proved. □

Proof of Theorem 2.2. Let, as before,

$$H_{n,A}^q f(x) \leq I\left(\frac{\pi}{n+1}\right) + J\left(\frac{\pi}{n+1}\right)$$

and

$$I\left(\frac{\pi}{n+1}\right) \leq \frac{n+1}{\pi} \int_0^{\frac{\pi}{n+1}} |\varphi_x(t)|\, dt = w_x\left(\frac{\pi}{n+1}\right)_{L^1}.$$

In the estimate of $J\left(\frac{\pi}{n+1}\right)$ we again use the Abel transformation and (2.2). Thus

$$\left(J\left(\frac{\pi}{n+1}\right)\right)^q \leq (K+1)\, a_{n,n} \sum_{\nu=0}^{n} \left| \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \varphi_x(t) \frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t} dt \right|^q$$

and, by inequality (3.2),

$$J\left(\frac{\pi}{n+1}\right) \leq K\left(a_{n,n}\right)^{\frac{1}{q}}\left\{\int_{\frac{\pi}{n+1}}^{\pi} \frac{\left|\varphi_x\left(t\right)\right|^p}{t^{1+p/q}} dt\right\}^{\frac{1}{p}}.$$

Integrating (2.3) by parts, with the assumption $f \in L^p\left(w_x\right)$, we obtain

$$J\left(\frac{\pi}{n+1}\right) \leq K\left(\left(n+1\right)a_{n,n}\right)^{\frac{1}{q}}\left\{\left(\frac{\pi}{n+1}\right)^{\frac{p}{q}}\int_{\frac{\pi}{n+1}}^{\pi} \frac{\left(w_x\left(t\right)\right)^p}{t^{1+p/q}} dt\right\}^{\frac{1}{p}}$$
$$= O\left(\left(\left(n+1\right)a_{n,n}\right)^{\frac{1}{q}}\frac{\pi}{n+1}H_x\left(\frac{\pi}{n+1}\right)\right)$$

as in the previous proof, with $\frac{\pi}{n+1}$ instead of $a_{n,n}$.

Finally, arguing as in [3, p.110], we can see that, for $j = 0, 1, \ldots, n-1$,

$$\left|a_{n,j} - a_{n,n}\right| \leq \left|\sum_{k=j}^{n-1}\left(a_{n,k} - a_{n,k+1}\right)\right| \leq \sum_{k=0}^{n-1}\left|a_{n,k} - a_{n,k+1}\right| \leq K a_{n,n} ,$$

whence

$$a_{n,j} \leq \left(K+1\right)a_{n,n}$$

and therefore

$$\left(K+1\right)\left(n+1\right)a_{n,n} \geq \sum_{j=0}^{n} a_{n,j} = 1.$$

This inequality implies that

$$J\left(\frac{\pi}{n+1}\right) = O\left(a_{n,n}H_x\left(\frac{\pi}{n+1}\right)\right)$$

and the proof of the first part of our statement is complete.

To prove of the second part of our assertion we have to estimate the term $I\left(\frac{\pi}{n+1}\right)$ once more.

Proceeding analogously to the proof of Theorem 2.1, with $a_{n,n}$ replaced by $\frac{\pi}{n+1}$, we obtain

$$I\left(\frac{\pi}{n+1}\right) = O\left(\frac{\pi}{n+1}H_x\left(\frac{\pi}{n+1}\right)\right).$$

By the inequality from the first part of our proof, the relation $\left(n+1\right)^{-1} = O\left(a_{n,n}\right)$ holds, whence the second statement follows. □

Proof of Theorem 2.3. As usual, let

$$H_{n,A}^q f\left(x\right) \leq I\left(a_{n,0}\right) + J\left(a_{n,0}\right).$$

Since $f \in L^p\left(w_x\right)$,by the same method as in the proof of Theorem 2.1, Lemma 3.1 and (2.3) yield

$$I\left(a_{n,0}\right) = O\left(a_{n,0}H_x\left(a_{n,0}\right)\right).$$

By the Abel transformation

$$\begin{aligned}(J(a_{n,0}))^{q} &\leq \sum_{k=0}^{n-1}|a_{n,k}-a_{n,k+1}|\sum_{\nu=0}^{k}\left|\frac{1}{\pi}\int_{a_{n,0}}^{\pi}\varphi_{x}(t)\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt\right|^{q}\\ &\quad + a_{n,n}\sum_{\nu=0}^{n}\left|\frac{1}{\pi}\int_{a_{n,0}}^{\pi}\varphi_{x}(t)\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt\right|^{q}\\ &\leq\left(\sum_{k=0}^{\infty}|a_{n,k}-a_{n,k+1}|+a_{n,n}\right)\sum_{\nu=0}^{\infty}\left|\frac{1}{\pi}\int_{a_{n,0}}^{\pi}\varphi_{x}(t)\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt\right|^{q}.\end{aligned}$$

Arguing as in [3, p.110], by (2.5), we have

$$|a_{n,n}-a_{n,0}|\leq\sum_{k=0}^{n-1}|a_{n,k}-a_{n,k+1}|\leq\sum_{k=0}^{\infty}|a_{n,k}-a_{n,k+1}|\leq Ka_{n,0},$$

whence $a_{n,n}\leq(K+1)\,a_{n,0}$ and therefore

$$\sum_{k=0}^{\infty}|a_{n,k}-a_{n,k+1}|+a_{n,n}\leq(2K+1)\,a_{n,0}$$

and

$$(J(a_{n,0}))^{q}\leq(2K+1)\,a_{n,0}\sum_{\nu=0}^{\infty}\left|\frac{1}{\pi}\int_{a_{n,0}}^{\pi}\varphi_{x}(t)\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt\right|^{q}.$$

Finally, by (3.2),

$$J(a_{n,0})\leq K(a_{n,0})^{\frac{1}{q}}\left\{\int_{a_{n,0}}^{\pi}\frac{|\varphi_{x}(t)|^{p}}{t^{1+p/q}}dt\right\}^{\frac{1}{p}}$$

and, by (2.3),

$$J(a_{n,0})=O\left(a_{n,0}H_{x}(a_{n,0})\right).$$

This completes of our proof. □

Proof of Theorem 2.4. We start as usual with the simple transformation

$$H_{n,A}^{q}f(x)\leq I\left(\frac{\pi}{n+1}\right)+J\left(\frac{\pi}{n+1}\right).$$

Similarly, as in the previous proofs, by (2.1) we have

$$I\left(\frac{\pi}{n+1}\right)\leq w_{x}f\left(\frac{\pi}{n+1}\right)_{L^{1}}.$$

We estimate the term J in the following way

$$\begin{aligned}\left(J\left(\frac{\pi}{n+1}\right)\right)^{q} &\leq \sum_{k=0}^{n-1}|a_{n,k}-a_{n,k+1}|\sum_{\nu=0}^{k}\left|\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\varphi_{x}(t)\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt\right|^{q}\\ &\quad + a_{n,n}\sum_{\nu=0}^{n}\left|\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\varphi_{x}(t)\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt\right|^{q}\\ &\leq\left(\sum_{k=0}^{\infty}|a_{n,k}-a_{n,k+1}|+a_{n,n}\right)\sum_{\nu=0}^{\infty}\left|\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\varphi_{x}(t)\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt\right|^{q}.\end{aligned}$$

From the assumption (2.5), arguing as before, we can see that

$$J\left(\frac{\pi}{n+1}\right)\leq K\left(a_{n,0}\sum_{\nu=0}^{\infty}\left|\frac{1}{\pi}\int_{\frac{\pi}{n+1}}^{\pi}\varphi_x(t)\frac{\sin\left(\nu+\frac{1}{2}\right)t}{2\sin\frac{1}{2}t}dt\right|^q\right)^{\frac{1}{q}}$$

and, by (3.2),

$$J\left(\frac{\pi}{n+1}\right)\leq K\left(a_{n,0}\right)^{\frac{1}{q}}\left\{\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|^p}{t^{1+p/q}}dt\right\}^{\frac{1}{p}}.$$

From (2.5), it follows that $a_{n,k}\leq(K+1)\,a_{n,0}$ for any $k\leq n$, and therefore

$$(K+1)(n+1)\,a_{n,0}\geq\sum_{k=0}^{n}a_{n,k}=1,$$

whence

$$\begin{aligned}J\left(\frac{\pi}{n+1}\right)&\leq K\left((n+1)\,a_{n,0}\right)^{\frac{1}{q}}\left\{\left(\frac{\pi}{n+1}\right)^{\frac{p}{q}}\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|^p}{t^{1+p/q}}dt\right\}^{\frac{1}{p}}.\\&\leq K(n+1)\,a_{n,0}\left\{\left(\frac{\pi}{n+1}\right)^{\frac{p}{q}}\int_{\frac{\pi}{n+1}}^{\pi}\frac{|\varphi_x(t)|^p}{t^{1+p/q}}dt\right\}^{\frac{1}{p}}.\end{aligned}$$

Since $f\in L^p(w_x)$, integrating by parts we obtain

$$\begin{aligned}J\left(\frac{\pi}{n+1}\right)&\leq K(n+1)\,a_{n,0}\frac{\pi}{n+1}H_x\left(\frac{\pi}{n+1}\right)\\&\leq Ka_{n,0}H_x\left(\frac{\pi}{n+1}\right)\end{aligned}$$

and the proof of the first part of our statement is complete.

To prove the second part, we have to estimate the term $I\left(\frac{\pi}{n+1}\right)$ once more.

Proceeding analogously to the proof of Theorem 2.1 we obtain

$$I\left(\frac{\pi}{n+1}\right)=O\left(\frac{\pi}{n+1}H_x\left(\frac{\pi}{n+1}\right)\right).$$

From the start of our proof we have $(n+1)^{-1}=O(a_{n,0})$, whence the second assertion also follows. □

Proof of Theorem 2.5. We begin with the inequality

$$\left\|H_{n,A}^q f(\cdot)\right\|_{X^{\tilde{p}}}\leq\left\|I(a_{n,n})\right\|_{X^{\tilde{p}}}+\left\|J(a_{n,n})\right\|_{X^{\tilde{p}}}.$$

By (2.1), Lemma 3.1 gives

$$\begin{aligned}\left\|I(a_{n,n})\right\|_{X^{\tilde{p}}}&\leq\int_0^{a_{n,n}}\frac{\left\|\varphi(t)\right\|_{X^{\tilde{p}}}}{2t}dt\\&\leq\int_0^{a_{n,n}}\frac{\omega f(t)_{X^{\tilde{p}}}}{2t}dt\\&=O\left(a_{n,n}H(a_{n,n})\right).\end{aligned}$$

As in the proof of Theorem 2.1,

$$\begin{aligned}\|J(a_{n,n})\|_{X^{\tilde{p}}} &\leq K(a_{n,n})^{\frac{1}{q}} \left\| \left\{ \int_{a_{n,n}}^{\pi} \frac{|\varphi(t)|^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}} \right\|_{X^{\tilde{p}}} \\ &\leq K(a_{n,n})^{\frac{1}{q}} \left\{ \int_{a_{n,n}}^{\pi} \frac{\|\varphi(t)\|_{X^{\tilde{p}}}^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}} \\ &\leq K(a_{n,n})^{\frac{1}{q}} \left\{ \int_{a_{n,n}}^{\pi} \frac{\omega f(t)_{X^{\tilde{p}}}}{t^{1+p/q}} dt \right\}^{\frac{1}{p}},\end{aligned}$$

whence, by (2.6),

$$\|J(a_{n,n})\|_{X^{\tilde{p}}} = O(a_{n,n} H(a_{n,n}))$$

holds and our result follows. □

Proof of Theorem 2.6. It is clear that

$$\left\|H_{n,A}^q f(\cdot)\right\|_{X^{\tilde{p}}} \leq \left\| I\left(\frac{\pi}{n+1}\right) \right\|_{X^{\tilde{p}}} + \left\| J\left(\frac{\pi}{n+1}\right) \right\|_{X^{\tilde{p}}}$$

and immediately

$$\left\| I\left(\frac{\pi}{n+1}\right) \right\|_{X^{\tilde{p}}} \leq \left\| w \cdot f\left(\frac{\pi}{n+1}\right)_1 \right\|_{X^{\tilde{p}}} \leq \omega f\left(\frac{\pi}{n+1}\right)_{X^{\tilde{p}}}$$

and

$$\begin{aligned}\left\| J\left(\frac{\pi}{n+1}\right) \right\|_{X^{\tilde{p}}} &\leq K(a_{n,n})^{\frac{1}{q}} \left\{ \int_{\frac{\pi}{n+1}}^{\pi} \frac{\|\varphi(t)\|_{X^{\tilde{p}}}^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}} \\ &\leq K((n+1)a_{n,n})^{\frac{1}{q}} \left\{ \frac{\pi}{n+1} \int_{\frac{\pi}{n+1}}^{\pi} \frac{\omega f(t)_{X^{\tilde{p}}}}{t^{1+p/q}} dt \right\}^{\frac{1}{p}} \\ &\leq K((n+1)a_{n,n})^{\frac{1}{q}} \frac{\pi}{n+1} H\left(\frac{\pi}{n+1}\right) \\ &\leq K(n+1) a_{n,n} \frac{\pi}{n+1} H\left(\frac{\pi}{n+1}\right) \\ &= O\left(a_{n,n} H\left(\frac{\pi}{n+1}\right)\right).\end{aligned}$$

Thus our first statement holds. The second one follows on using a similar process to that in the proof of Theorem 2.1. We have to only use the estimates obtained in the proof of Theorem 2.5, with $\frac{\pi}{n+1}$ instead of $a_{n,n}$,and the relation

$$(n+1)^{-1} = O(a_{n,n}).$$

□

Proof of Theorem 2.7. As in the proof of Theorem 2.5, we have

$$\left\|H_{n,A}^q f(\cdot)\right\|_{X^{\tilde{p}}} \leq \|I(a_{n,0})\|_{X^{\tilde{p}}} + \|J(a_{n,0})\|_{X^{\tilde{p}}}$$

and

$$\|I(a_{n,0})\|_{X^{\tilde{p}}} = O(a_{n,0}H(a_{n,0}))\ .$$

Also, from the proof of Theorem 2.3,

$$\begin{aligned}\|J(a_{n,0})\|_{X^{\tilde{p}}} &\leq K(a_{n,0})^{\frac{1}{q}} \left\| \left\{ \int_{a_{n,0}}^{\pi} \frac{|\varphi_{\cdot}(t)|^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}} \right\|_{X^{\tilde{p}}} \\ &\leq K(a_{n,0})^{\frac{1}{q}} \left\{ \int_{a_{n,0}}^{\pi} \frac{\omega f(t)_{X^{\tilde{p}}}}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}\end{aligned}$$

and, by (2.6),

$$\|J(a_{n,0})\|_{X^{\tilde{p}}} = O(a_{n,0}H(a_{n,0}))\ .$$

Thus our result is proved. □

Proof of Theorem 2.8. We recall, as in the previous proof, that

$$\left\|H_{n,A}^{q}f(\cdot)\right\|_{X^{\tilde{p}}} \leq \left\|I\left(\frac{\pi}{n+1}\right)\right\|_{X^{\tilde{p}}} + \left\|J\left(\frac{\pi}{n+1}\right)\right\|_{X^{\tilde{p}}}$$

and

$$\left\|I\left(\frac{\pi}{n+1}\right)\right\|_{X^{\tilde{p}}} \leq \omega f\left(\frac{\pi}{n+1}\right)_{X^{\tilde{p}}}.$$

We apply a similar method as that used in the proof of Theorem 2.4 to obtain an estimate for the quantity $\left\|J\left(\frac{\pi}{n+1}\right)\right\|_{X^{\tilde{p}}}$,

$$\begin{aligned}\left\|J\left(\frac{\pi}{n+1}\right)\right\|_{X^{\tilde{p}}} &\leq K(n+1)a_{n,0} \left\| \left\{ \left(\frac{\pi}{n+1}\right)^{\frac{p}{q}} \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\varphi_x(t)|^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}} \right\|_{X^{\tilde{p}}} \\ &\leq K(n+1)a_{n,0} \left\{ \left(\frac{\pi}{n+1}\right)^{\frac{p}{q}} \int_{\frac{\pi}{n+1}}^{\pi} \frac{\|\varphi_{\cdot}(t)\|_{X^{\tilde{p}}}^p}{t^{1+p/q}} dt \right\}^{\frac{1}{p}} \\ &\leq K(n+1)a_{n,0} \left\{ \left(\frac{\pi}{n+1}\right)^{\frac{p}{q}} \int_{\frac{\pi}{n+1}}^{\pi} \frac{\omega f(t)_{X^{\tilde{p}}}}{t^{1+p/q}} dt \right\}^{\frac{1}{p}}\end{aligned}$$

and, by (2.6),

$$\begin{aligned}\left\|J\left(\frac{\pi}{n+1}\right)\right\|_{X^{\tilde{p}}} &\leq K(n+1)a_{n,0}\frac{\pi}{n+1}H\left(\frac{\pi}{n+1}\right) \\ &\leq K a_{n,0} H\left(\frac{\pi}{n+1}\right).\end{aligned}$$

Thus the proof of the first part of our statement is complete.

To prove of the second part, we follow the line of the proof of Theorem 2.6. □

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