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ON MULTIVARIATE OSTROWSKI TYPE INEQUALITIES

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ABSTRACT. In the present paper we establish new multivariate Ostrowski type inequalities by using fairly elementary analysis.

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1. Introduction

The following inequality is well known in the literature as Ostrowski's integral inequality (see [5, p. 469]).

Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \le \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)\,\|f'\|_\infty ,$$

for all $x \in [a,b]$.

Many generalisations, extensions and variants of this inequality have appeared in the literature, see [1] – [7] and the references given therein.

The main aim of this paper is to establish new inequalities similar to that of Ostrowski's inequality involving functions of many independent variables and their first order partial derivatives. The analysis used in the proof is elementary and our results provide new estimates on these types of inequalities.

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2. Statement of Results

In what follows, $\mathbb{R}$ denotes the set of real nummbers, $\mathbb{R}^n$ the $n-$dimensional Euclidean space. Let $D = \{(x_1, \dots, x_n) : a_i < x_i < b_i \quad (i = 1, \dots, n)\}$ and $\bar{D}$ be the closure of D. For a function $u(x) : \mathbb{R}^n \to \mathbb{R}$, we denote the first order partial derivatives by $\frac{\partial u(x)}{\partial x_i}$ $(i = 1, \dots, n)$ and $\int_D u(x)\,dx$ the $n-$fold integral $\int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(x_1, \dots, x_n)\,dx_1 \dots dx_n$.

Our main results are established in the following theorems.

Theorem 2.1. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be continuous functions on $\bar{D}$ and differentiable on D whose derivatives $\frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i}$ are bounded, i.e.,*

$$\left\| \frac{\partial f}{\partial x_i} \right\|_\infty = \sup_{x \in D} \left| \frac{\partial f(x)}{\partial x_i} \right| < \infty,$$
$$\left\| \frac{\partial g}{\partial x_i} \right\|_\infty = \sup_{x \in D} \left| \frac{\partial g(x)}{\partial x_i} \right| < \infty.$$

Let the function $w(x)$ be defined, nonnegative, integrable for every $x \in D$ and $\int_D w(y)\,dy > 0$. Then for every $x \in \bar{D}$,

$$\left| f(x) g(x) - \frac{1}{2M} g(x) \int_D f(y)\,dy - \frac{1}{2M} f(x) \int_D g(y)\,dy \right| \leq \frac{1}{2M} \sum_{i=1}^n \left[|g(x)| \left\| \frac{\partial f}{\partial x_i} \right\|_\infty + |f(x)| \left\| \frac{\partial g}{\partial x_i} \right\|_\infty \right] E_i(x), \tag{2.1}$$

$$\left| f(x) g(x) - \left[\frac{g(x) \int_D w(y) f(y)\,dy + f(x) \int_D w(y) g(y)\,dy}{2 \int_D w(y)\,dy} \right] \right| \leq \frac{\int_D w(y) \sum_{i=1}^n \left[|g(x)| \left\| \frac{\partial f}{\partial x_i} \right\|_\infty + |f(x)| \left\| \frac{\partial g}{\partial x_i} \right\|_\infty \right] |x_i - y_i|\,dy}{2 \int_D w(y)\,dy}, \tag{2.2}$$

where $M = \operatorname{mes} D = \prod_{i=1}^n (b_i - a_i)$, $dy = dy_1 \dots dy_n$ and $E_i(x) = \int_D |x_i - y_i|\,dy$.

Remark 2.2. If we take $g(x) = 1$ and hence $\frac{\partial g}{\partial x_i} = 0$ in Theorem 2.1, then the inequalities (2.1) and (2.2) reduces to the inequalities established by Milovanović in [3, Theorems 2 and 3] which in turn are the further generalisations of the well known Ostrowski's inequality.

Theorem 2.3. *Let $f, g, \frac{\partial f}{\partial x_i}, \frac{\partial g}{\partial x_i}$ be as in Theorem 2.1. Then for every $x \in \bar{D}$,*

$$\left| f(x) g(x) - f(x) \left(\frac{1}{M} \int_D g(y)\,dy \right) - g(x) \left(\frac{1}{M} \int_D f(y)\,dy \right) + \frac{1}{M} \int_D f(y) g(y)\,dy \right| \leq \frac{1}{M} \int_D \left[\left(\sum_{i=1}^n \left\| \frac{\partial f}{\partial x_i} \right\|_\infty |x_i - y_i| \right) \left(\sum_{i=1}^n \left\| \frac{\partial g}{\partial x_i} \right\|_\infty |x_i - y_i| \right) \right] dy, \tag{2.3}$$

$$\left|f(x)g(x) - f(x)\left(\frac{1}{M}\int_D g(y)\,dy\right) - g(x)\left(\frac{1}{M}\int_D f(y)\,dy\right) + \frac{1}{M^2}\left(\int_D f(y)\,dy\right)\left(\int_D g(y)\,dy\right)\right| \le \frac{1}{M^2}\left(\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty E_i(x)\right)\left(\sum_{i=1}^{n}\left\|\frac{\partial g}{\partial x_i}\right\|_\infty E_i(x)\right), \tag{2.4}$$

where M, dy and $E_i(x)$ are as defined in Theorem 2.1.

Remark 2.4. We note that in [1] Anastassiou has used a slightly different technique to establish multivariate Ostrowski type inequalities. However, the inequalities established in (2.3) and (2.4) are different from those given in [1] and our proofs are extremely simple. For an $n-$dimensional version of Ostrowski's inequality for mappings of Hölder type, see [2].

3. PROOF OF THEOREM 2.1

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ $\left(x \in \bar{D},\ y \in D\right)$. From the $n-$dimensional version of the mean value theorem, we have (see [8, p. 174] or [4, p. 121])

$$f(x) - f(y) = \sum_{i=1}^{n}\frac{\partial f(c)}{\partial x_i}(x_i - y_i), \tag{3.1}$$

$$g(x) - g(y) = \sum_{i=1}^{n}\frac{\partial g(c)}{\partial x_i}(x_i - y_i), \tag{3.2}$$

where $c = (y_1 + \alpha(x_1 - y_1), \ldots, y_n + \alpha(x_n - y_n))$ $(0 < \alpha < 1)$.

Multiplying both sides of (3.1) and (3.2) by $g(x)$ and $f(x)$ respectively and adding, we get

$$2f(x)g(x) - g(x)f(y) - f(x)g(y) = g(x)\sum_{i=1}^{n}\frac{\partial f(c)}{\partial x_i}(x_i - y_i) + f(x)\sum_{i=1}^{n}\frac{\partial g(c)}{\partial x_i}(x_i - y_i). \tag{3.3}$$

Integrating both sides of (3.3) with respect to y over D, using the fact that $\operatorname{mes} D > 0$ and rewriting, we have

$$f(x)g(x) - \frac{1}{2M}g(x)\int_D f(y)\,dy - \frac{1}{2M}f(x)\int_D g(y)\,dy = \frac{1}{2M}\left[g(x)\int_D\sum_{i=1}^{n}\frac{\partial f(c)}{\partial x_i}(x_i - y_i)\,dy + f(x)\int_D\sum_{i=1}^{n}\frac{\partial g(c)}{\partial x_i}(x_i - y_i)\,dy\right]. \tag{3.4}$$

From (3.4) and using the properties of modulus we have

$$\begin{aligned}\left|f(x)g(x) - \frac{1}{2M}g(x)\int_D f(y)\,dy - \frac{1}{2M}f(x)\int_D g(y)\,dy\right| &\le \frac{1}{2M}\left[|g(x)|\int_D\sum_{i=1}^{n}\left|\frac{\partial f(c)}{\partial x_i}\right||x_i - y_i|\,dy + |f(x)|\int_D\sum_{i=1}^{n}\left|\frac{\partial g(c)}{\partial x_i}\right||x_i - y_i|\,dy\right] \\ &\le \frac{1}{2M}\sum_{i=1}^{n}\left[|g(x)|\left\|\frac{\partial f}{\partial x_i}\right\|_\infty + |f(x)|\left\|\frac{\partial g}{\partial x_i}\right\|_\infty\right]E_i(x).\end{aligned}$$

The proof of the inequality (2.1) is complete.

Multiplying both sides of (3.3) by $w(y)$ and integrating the resulting identity with respect to y on D and following the proof of inequality (2.1), we get the desired inequality in (2.2).

4. PROOF OF THEOREM 2.3

From the hypotheses, as in the proof of Theorem 2.1, the identities (3.1) and (3.2) hold. Multiplying the left and right sides of (3.1) and (3.2) we get

$$f(x)g(x)-f(x)g(y)-g(x)f(y)+f(y)g(y) = \left[\sum_{i=1}^{n}\frac{\partial f(c)}{\partial x_i}(x_i-y_i)\right]\left[\sum_{i=1}^{n}\frac{\partial g(c)}{\partial x_i}(x_i-y_i)\right]. \tag{4.1}$$

Integrating both sides of (4.1) with respect to y on D and rewriting, we have

$$\begin{aligned} f(x)g(x)&-f(x)\left(\frac{1}{M}\int_D g(y)\,dy\right) - g(x)\left(\frac{1}{M}\int_D f(y)\,dy\right)+\frac{1}{M}\int_D f(y)g(y)\,dy \\ &= \frac{1}{M}\int_D \left[\sum_{i=1}^{n}\frac{\partial f(c)}{\partial x_i}(x_i-y_i)\right]\left[\sum_{i=1}^{n}\frac{\partial g(c)}{\partial x_i}(x_i-y_i)\right]dy. \end{aligned} \tag{4.2}$$

From (4.2) and using the properties of the modulus, we have

$$\begin{aligned} &\left| f(x)g(x)-f(x)\left(\frac{1}{M}\int_D g(y)\,dy\right) -g(x)\left(\frac{1}{M}\int_D f(y)\,dy\right)+\frac{1}{M}\int_D f(y)g(y)\,dy\right| \\ &\leq \frac{1}{M}\int_D \left[\sum_{i=1}^{n}\left|\frac{\partial f(c)}{\partial x_i}\right||x_i-y_i|\right]\left[\sum_{i=1}^{n}\left|\frac{\partial g(c)}{\partial x_i}\right||x_i-y_i|\right]dy \\ &\leq \frac{1}{M}\int_D \left[\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty|x_i-y_i|\right]\left[\sum_{i=1}^{n}\left\|\frac{\partial g}{\partial x_i}\right\|_\infty|x_i-y_i|\right]dy, \end{aligned}$$

which is the required inequality in (2.3).

Integrating both sides of (3.1) and (3.2) with respect to y over D and rewriting, we get

$$f(x)-\frac{1}{M}\int_D f(y)\,dy = \frac{1}{M}\int_D \sum_{i=1}^{n}\frac{\partial f(c)}{\partial x_i}(x_i-y_i)\,dy, \tag{4.3}$$

and

$$g(x)-\frac{1}{M}\int_D g(y)\,dy = \frac{1}{M}\int_D \sum_{i=1}^{n}\frac{\partial g(c)}{\partial x_i}(x_i-y_i)\,dy, \tag{4.4}$$

respectively. Multiplying the left and right sides of (4.3) and (4.4) we get

$$\begin{aligned}(4.5)\quad & f(x)\,g(x) - f(x)\left(\frac{1}{M}\int_D g(y)\,dy\right) \\ & - g(x)\left(\frac{1}{M}\int_D f(y)\,dy\right) + \frac{1}{M^2}\left(\int_D f(y)\,dy\right)\left(\int_D g(y)\,dy\right) \\ & = \frac{1}{M^2}\left(\int_D \sum_{i=1}^{n}\frac{\partial f(c)}{\partial x_i}(x_i-y_i)\,dy\right)\left(\int_D \sum_{i=1}^{n}\frac{\partial g(c)}{\partial x_i}(x_i-y_i)\,dy\right).\end{aligned}$$

From (4.5) and using the properties of the modulus we have

$$\begin{aligned}&\Bigg| f(x)\,g(x) - f(x)\left(\frac{1}{M}\int_D g(y)\,dy\right) \\ &\qquad - g(x)\left(\frac{1}{M}\int_D f(y)\,dy\right) + \frac{1}{M^2}\left(\int_D f(y)\,dy\right)\left(\int_D g(y)\,dy\right)\Bigg| \\ &\le \frac{1}{M^2}\left(\int_D \sum_{i=1}^{n}\left|\frac{\partial f(c)}{\partial x_i}\right| |x_i-y_i|\,dy\right)\left(\int_D \left|\frac{\partial g(c)}{\partial x_i}\right| |x_i-y_i|\,dy\right) \\ &\le \frac{1}{M^2}\left(\sum_{i=1}^{n}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty E_i(x)\right)\left(\sum_{i=1}^{n}\left\|\frac{\partial g}{\partial x_i}\right\|_\infty E_i(x)\right).\end{aligned}$$

This is the desired inequality in (2.4) and the proof is complete.

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