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BOUNDED LINEAR OPERATORS IN PROBABILISTIC NORMED SPACE

IQBAL H. JEBRIL AND RADHI IBRAHIM M. ALI

University of Al al-BAYT,
Department of Mathematics,
P.O.Box 130040,
Mafrq 25113, Jordan.
EMail: igbal501@yahoo.com

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Abstract

The notion of a probabilistic metric space was introduced by Menger in 1942. The notion of a probabilistic normed space was introduced in 1993. The aim of this paper is to give a necessary condition to get bounded linear operators in probabilistic normed space.

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Key words: Probabilistic Normed Space, Bounded Linear Operators.

It is a pleasure to thank C. Alsina and C. Sempì for sending us the references [1, 3, 9].

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1. Introduction

The purpose of this paper is to present a definition of bounded linear operators which is based on the new definition of a probabilistic normed space. This definition is sufficiently general to encompass the most important contraction function in probabilistic normed space. The concepts used are those of [1], [2] and [9].

A *distribution function* (briefly, a d.f.) is a function F from the extended real line $\bar{\mathbb{R}} = [-\infty, +\infty]$ into the unit interval $I = [0, 1]$ that is nondecreasing and satisfies $F(-\infty) = 0$, $F(+\infty) = 1$. We normalize all d.f.'s to be left-continuous on the unextended real line $\mathbb{R} = (-\infty, +\infty)$. For any $a \geq 0$, ε_a is the d.f. defined by

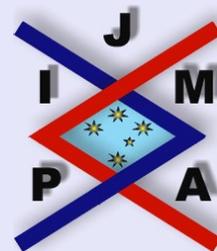
$$(1.1) \quad \varepsilon_a(x) = \begin{cases} 0, & \text{if } x \leq a \\ 1, & \text{if } x > a, \end{cases}$$

The set of all the d.f.s will be denoted by Δ and the subset of those d.f.s called positive d.f.s. such that $F(0) = 0$, by Δ^+ .

By setting $F \leq G$ whenever $F(x) \leq G(x)$ for all x in \mathbb{R} , the maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

A *triangle function* is a binary operation on Δ^+ , namely a function $\tau : \Delta^+ \times \Delta^+ \rightarrow \Delta^+$ that is associative, commutative, nondecreasing and which has ε_0 as



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unit, that is, for all $F, G, H \in \Delta^+$, we have

$$\begin{aligned}\tau(\tau(F, G), H) &= \tau(F, \tau(G, H)), \\ \tau(F, G) &= \tau(G, F), \\ \tau(F, H) &\leq \tau(G, H), \quad \text{if } F \leq G, \\ \tau(F, \varepsilon_0) &= F.\end{aligned}$$

Continuity of a triangle function means continuity with respect to the topology of weak convergence in Δ^+ .

Typical continuous triangle functions are convolution and the operations τ_T and τ_{T^*} , which are, respectively, given by

$$(1.2) \quad \tau_T(F, G)(x) = \sup_{s+t=x} T(F(s), G(t)),$$

and

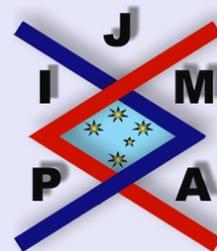
$$(1.3) \quad \tau_{T^*}(F, G)(x) = \inf_{s+t=x} T^*(F(s), G(t)),$$

for all F, G in Δ^+ and all x in \mathbb{R} [9, Sections 7.2 and 7.3], here T is a continuous t -norm, i.e. a continuous binary operation on $[0, 1]$ that is associative, commutative, nondecreasing and has 1 as identity; T^* is a continuous t -conorm, namely a continuous binary operation on $[0, 1]$ that is related to continuous t -norm through

$$(1.4) \quad T^*(x, y) = 1 - T(1 - x, 1 - y).$$

It follows without difficulty from (1.1)–(1.4) that

$$\tau_T(\varepsilon_a, \varepsilon_b) = \varepsilon_{a+b} = \tau_{T^*}(\varepsilon_a, \tau_b),$$



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for any continuous t -norm T , any continuous t -conorm T^* and any $a, b \geq 0$.

The most important t -norms are the functions W , $Prod$, and M which are defined, respectively, by

$$\begin{aligned} W(a, b) &= \max(a + b - 1, 0), \\ prod(a, b) &= a \cdot b, \\ M(a, b) &= \min(a, b). \end{aligned}$$

Their corresponding t -norms are given, respectively, by

$$\begin{aligned} W^*(a, b) &= \min(a + b, 1), \\ prod^*(a, b) &= a + b - a \cdot b, \\ M^*(a, b) &= \max(a, b). \end{aligned}$$

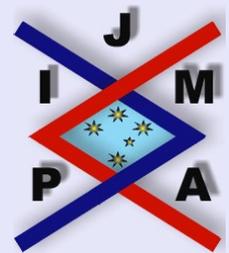
Definition 1.1. A probabilistic metric (briefly PM) space is a triple (S, f, τ) , where S is a nonempty set, τ is a triangle function, and f is a mapping from $S \times S$ into Δ^+ such that, if F_{pq} denoted the value of f at the pair (p, q) , the following hold for all p, q, r in S :

(PM1) $F_{pq} = \varepsilon_0$ if and only if $p = q$.

(PM2) $F_{pq} = F_{qp}$.

(PM3) $F_{pr} \geq \tau(F_{pq}, F_{qr})$.

Definition 1.2. A probabilistic normed space is a quadruple (V, ν, τ, τ^*) , where V is a real vector space, τ and τ^* are continuous triangle functions, and ν is a mapping from V into Δ^+ such that, for all p, q in V , the following conditions hold:



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(PN1) $\nu_p = \varepsilon_0$ if and only if $p = \theta$, θ being the null vector in V ;

(PN2) $\nu_{-p} = \nu_p$;

(PN3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$

(PN4) $\nu_p \leq \tau^*(\nu_{\alpha p}, \nu_{(1-\alpha)p})$ for all α in $[0, 1]$.

If, instead of (PN1), we only have $\nu_\theta = \varepsilon_\theta$, then we shall speak of a *Probabilistic Pseudo Normed Space*, briefly a PPN space. If the inequality (PN4) is replaced by the equality $V_p = \tau_M(\nu_{\alpha p}, \nu_{(1-\alpha)p})$, then the PN space is called a *Serstnev space*. The pair is said to be a *Probabilistic Seminormed Space* (briefly PSN space) if $\nu : V \rightarrow \Delta^+$ satisfies (PN1) and (PN2).

Definition 1.3. A PSN (V, ν) space is said to be equilateral if there is a d.f. $F \in \Delta^+$ different from ε_0 and from ε_∞ , such that, for every $p \neq \theta$, $\nu_p = F$. Therefore, every equilateral PSN space (V, ν) is a PN space under $\tau = M$ and $\tau^* = M$ where is the triangle function defined for $G, H \in \Delta^+$ by

$$M(G, H)(x) = \min\{G(x), H(x)\} \quad (x \in [0, \infty]).$$

An equilateral PN space will be denoted by (V, F, M) .

Definition 1.4. Let $(V, \|\cdot\|)$ be a normed space and let $G \in \Delta^+$ be different from ε_0 and ε_∞ ; define $\nu : V \rightarrow \Delta^+$ by $\nu_\theta = \varepsilon_0$ and

$$\nu_p(t) = G\left(\frac{t}{\|p\|^\alpha}\right) \quad (p \neq \theta, t > 0),$$

where $\alpha \geq 0$. Then the pair (V, ν) will be called the α -simple space generated by $(V, \|\cdot\|)$ and by G .

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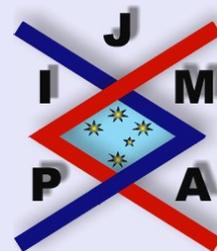
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The α -simple space generated by $(V, \|\cdot\|)$ and by G is immediately seen to be a PSN space; it will be denoted by $(V, \|\cdot\|, G; \alpha)$.

Definition 1.5. There is a natural topology in PN space (V, ν, τ, τ^*) , called the strong topology; it is defined by the neighborhoods,

$$N_p(t) = \{q \in V : \nu_{q-p}(t) > 1 - t\} = \{q \in d_L(\nu_{q-p}, \varepsilon_0) < t\},$$

where $t > 0$. Here d_L is the modified Levy metric ([9]).



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2. Bounded Linear Operators in Probabilistic Normed Spaces

In 1999, B. Guillen, J. Lallena and C. Sempì [3] gave the following definition of bounded set in PN space.

Definition 2.1. Let A be a nonempty set in PN space (V, ν, τ, τ^*) . Then

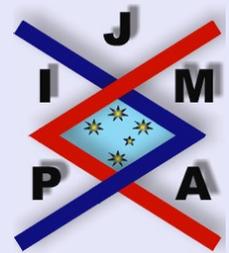
- (a) A is certainly bounded if, and only if, $\varphi_A(x_0) = 1$ for some $x_0 \in (0, +\infty)$;
- (b) A is perhaps bounded if, and only if, $\varphi_A(x_0) < 1$ for every $x_0 \in (0, +\infty)$ and $l^- \varphi_A(+\infty) = 1$;
- (c) A is perhaps unbounded if, and only if, $l^- \varphi_A(+\infty) \in (0, 1)$;
- (d) A is certainly unbounded if, and only if, $l^- \varphi_A(+\infty) = 0$; i.e., $\varphi_A(x) = 0$;

where $\varphi_A(x) = \inf \{ \nu_p(x) : P \in A \}$ and $l^- \varphi_A(x) = \lim_{t \rightarrow x^-} \varphi_A(t)$.

Moreover, A will be said to be D -bounded if either (a) or (b) holds.

Definition 2.2. Let (V, ν, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be PN spaces. A linear map $T : V \rightarrow V'$ is said to be

- (a) Certainly bounded if every certainly bounded set A of the space (V, ν, τ, τ^*) has, as image by T a certainly bounded set TA of the space $(V', \mu, \sigma, \sigma^*)$, i.e., if there exists $x_0 \in (0, +\infty)$ such that $\nu_p(x_0) = 1$ for all $p \in A$, then there exists $x_1 \in (0, +\infty)$ such that $\mu_{Tp}(x_1) = 1$ for all $p \in A$.



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(b) *Bounded if it maps every D -bounded set of V into a D -bounded set of V' , i.e., if, and only if, it satisfies the implication,*

$$\lim_{x \rightarrow +\infty} \varphi_A(x) = 1 \Rightarrow \lim_{x \rightarrow +\infty} \varphi_{TA}(x) = 1,$$

for every nonempty subset A of V .

(c) *Strongly \mathbf{B} -bounded if there exists a constant $k > 0$ such that, for every $p \in V$ and for every $x > 0$, $\mu_{Tp}(x) \geq \nu_p\left(\frac{x}{k}\right)$, or equivalently if there exists a constant $h > 0$ such that, for every $p \in V$ and for every $x > 0$,*

$$\mu_{Tp}(hx) \geq \nu_p(x).$$

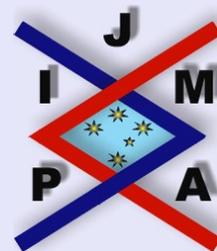
(d) *Strongly \mathbf{C} -bounded if there exists a constant $h \in (0, 1)$ such that, for every $p \in V$ and for every $x > 0$,*

$$\nu_p(x) > 1 - x \Rightarrow \mu_{Tp}(hx) > 1 - hx.$$

Remark 2.1. *The identity map I between PN space (V, ν, τ, τ^*) into itself is strongly \mathbf{C} -bounded. Also, all linear contraction mappings, according to the definition of [7, Section 1], are strongly \mathbf{C} -bounded, i.e for every $p \in V$ and for every $x > 0$ if the condition $\nu_p(x) > 1 - x$ is satisfied then*

$$\nu_{Ip}(hx) = \nu_p(hx) > 1 - hx.$$

But we note that when $k = 1$ then the identity map I between PN space (V, ν, τ, τ^*) into itself is a strongly \mathbf{B} -bounded operator. Also, all linear contraction mappings, according to the definition of [9, Section 12.6], are strongly \mathbf{B} -bounded.



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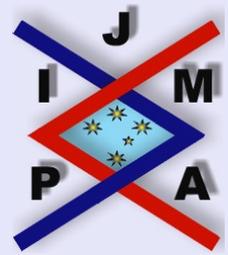


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In [3] B. Guillen, J. Lallena and C. Sempì present the following, every strongly **B**-bounded operator is also certainly bounded and every strongly **B**-bounded operator is also bounded. But the converses need not to be true.

Now we are going to prove that in the Definition 2.2, the notions of strongly **C**-bounded operator, certainly bounded, bounded and strongly **B**-bounded do not imply each other.

In the following example we will introduce a strongly **C**-bounded operator, which is not strongly **B**-bounded, not bounded nor certainly bounded.

Example 2.1. Let V be a vector space and let $\nu_\theta = \mu_\theta = \varepsilon_0$, while, if $p, q \neq \theta$ then, for every $p, q \in V$ and $x \in \mathbb{R}$, if

$$\nu_p(x) = \begin{cases} 0, & x \leq 1 \\ 1, & x > 1 \end{cases} \quad \mu_p(x) = \begin{cases} \frac{1}{3}, & x \leq 1 \\ \frac{9}{10}, & 1 < x < \infty \\ 1, & x = \infty \end{cases}$$

and if

$$\tau(\nu_p(x), \nu_q(y)) = \tau^*(\nu_p(x), \nu_q(y)) = \min(\nu_p(x), \nu_q(x)),$$

$$\sigma(\mu_p(x), \mu_q(y)) = \sigma^*(\mu_p(x), \mu_q(y)) = \min(\mu_p(x), \mu_q(x)),$$

then (V, ν, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ are equilateral PN spaces by Definition 1.3. Now let $I : (V, \nu, \tau, \tau^*) \rightarrow (V, \mu, \sigma, \sigma^*)$ be the identity operator, then I is strongly **C**-bounded but I is not strongly **B**-bounded, bounded and certainly bounded, it is clear that I is not certainly bounded and is not bounded. I is not

strongly **B**-bounded, because for every $k > 0$ and for $x = \max \left\{ 2, \frac{1}{k} \right\}$,

$$\mu_{I_p}(kx) = \frac{9}{10} < 1 = \nu_p(x).$$

But I is strongly **C**-bounded, because for every $p > 0$ and for every $x > 0$, this condition $\nu_p(x) > 1 - x$ is satisfied only if $x > 1$ now if $h = \frac{7}{10}x$ then

$$\mu_{I_p}(hx) = \mu_{I_p}\left(\frac{7}{10x}x\right) = \mu_p\left(\frac{7}{10}\right) = \frac{1}{3} > \frac{3}{10} = 1 - \frac{7}{10} = 1 - \left(\frac{7}{10x}\right)x.$$

Remark 2.2. We have noted in the above example that there is an operator, which is strongly **C**-bounded, but it is not strongly **B**-bounded. Moreover we are going to give an operator, which is strongly **B**-bounded, but it is not strongly **C**-bounded.

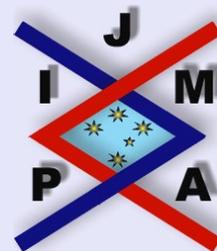
Definition 2.3. Let (V, ν, τ, τ^*) be PN space then we defined

$$B(p) = \inf \left\{ h \in \mathbb{R} : \nu_p(h^+) > 1 - h \right\}.$$

Lemma 2.1. Let $T : (V, \nu, \tau, \tau^*) \rightarrow (V', \mu, \sigma, \sigma^*)$ be a strongly **B**-bounded linear operator, for every p in V and let μ_{T_p} be strictly increasing on $[0, 1]$, then $B(T_p) < B(p)$, $\forall p \in V$.

Proof. Let $\eta \in \left(0, \frac{1-\gamma}{\gamma}B(p)\right)$, where $\gamma \in (0, 1)$. Then $B(p) > \gamma[B(p) + \eta]$ and so

$$\mu_{T_p}(B(p)) > \mu_{T_p}(\gamma[B(p) + \eta]),$$



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and where μ_{T_p} is strictly increasing on $[0, 1]$, then

$$\mu_{T_p}(\gamma[B(p) + \eta]) \geq \nu_p(B(p) + \eta) \geq \nu_p(B(p)^+) > 1 - B(p),$$

we conclude that

$$B(T_p) = \inf \{B(p) : \mu_{T_p}(B(p)^+) > 1 - B(p)\},$$

so $B(T_p) < B(p)$, $\forall p \in V$. □

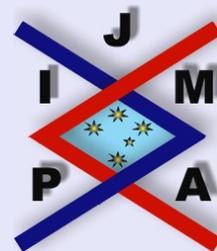
Theorem 2.2. *Let $T : (V, \nu, \tau, \tau^*) \rightarrow (V', \mu, \sigma, \sigma^*)$ be a strongly **B**-bounded linear operator, and let μ_{T_p} be strictly increasing on $[0, 1]$, then T is a strongly **C**-bounded linear operator.*

Proof. Let T be a strictly **B**-bounded operator. Since, by Lemma 2.1, $B(T_p) < B(p)$, $\forall p \in V$ there exist $\gamma_p \in (0, 1)$ such that $B(T_p) < \gamma_p B(p)$.

It means that

$$\begin{aligned} \inf \{h \in \mathbb{R} : \mu_{T_p}(h^+) > 1 - h\} &\leq \gamma \inf \{h \in \mathbb{R} : \nu_p(h^+) > 1 - h\} \\ &= \inf \{\gamma h \in \mathbb{R} : \nu_p(h^+) > 1 - h\} \\ &= \inf \left\{ h \in \mathbb{R} : \nu_p\left(\frac{h^+}{\gamma}\right) > 1 - \frac{h}{\gamma} \right\}. \end{aligned}$$

We conclude that $\nu_p\left(\frac{h}{\gamma}\right) > 1 - \left(\frac{h}{\gamma}\right) \implies \mu_{T_p}(h) > 1 - h$. Now if $x = \frac{h}{\gamma}$ then $\nu_p(x) > 1 - x \implies \mu_{T_p}(xh) > 1 - xh$, so T is a strongly **C**-bounded operator. □



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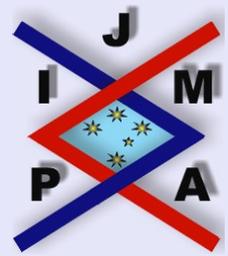


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Remark 2.3. From Theorem 2.2 we have noted that under some additional condition every a strongly **B**-bounded operator is a strongly **C**-bounded operator. But in general, it is not true.

Example 2.2. Let $V = V' = \mathbb{R}$ and $v_0 = \mu_0 = \varepsilon_0$, while, if $p \neq 0$, then, for $x > 0$, let $v_p(x) = G\left(\frac{x}{|p|}\right)$, $\mu_p(x) = U\left(\frac{x}{|p|}\right)$, where

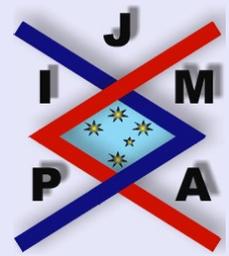
$$G(x) = \begin{cases} \frac{1}{2}, & 0 < x \leq 2, \\ 1, & 2 < x \leq +\infty, \end{cases} \quad U(x) = \begin{cases} \frac{1}{2}, & 0 < x \leq \frac{3}{2}, \\ 1, & \frac{3}{2} < x \leq +\infty \end{cases}.$$

Consider now the identity map $I : (\mathbb{R}, |\cdot|, G, \mu) \rightarrow (\mathbb{R}, |\cdot|, G, \mu)$. Now

(a) I is a strongly **B**-bounded operator, such that for every $p \in \mathbb{R}$ and every $x > 0$ then

$$\begin{aligned} \mu_{Ip}\left(\frac{3}{4}x\right) &= \mu_p\left(\frac{3}{4}x\right) \\ &= U\left(\frac{3x}{4|p|}\right) \\ &= \begin{cases} \frac{1}{2}, & 0 < x \leq 2|p|, \\ 1, & 2|p| < x \leq +\infty, \end{cases} = G\left(\frac{x}{|p|}\right) = v_p(x). \end{aligned}$$

(b) I is not a strongly **C**-bounded operator, such that for every $h \in (0, 1)$, let $x = \frac{3}{8h}$, $p = \frac{1}{4}$. If $x > 2|p|$ then the condition $v_p(x) > 1 - x$ will be



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satisfied, but we note that

$$\begin{aligned} \mu_{Ip}(hx) &= \mu_p(hx) \\ &= U\left(\frac{hx}{|p|}\right) \\ &= U\left(\frac{3}{2}\right) = \frac{1}{2} < \frac{5}{8} = 1 - h\left(\frac{3}{8h}\right) = 1 - hx. \end{aligned}$$

Now we introduce the relation between the strongly **B**-bounded and strongly **C**-bounded operators with boundedness in normed space.

Theorem 2.3. *Let G be strictly increasing on $[0, 1]$, then $T : (V, \|\cdot\|, G, \alpha) \rightarrow (V', \|\cdot\|, G, \alpha)$ is a strongly **B**-bounded operator if, and only if, T is a bounded linear operator in normed space.*

Proof. Let $k > 0$ and $x > 0$. Then for every $p \in V$

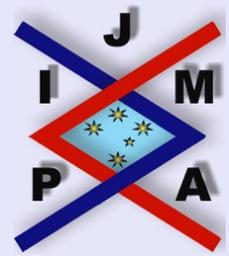
$$G\left(\frac{kx}{\|T_p\|^\alpha}\right) = \mu_{T_p}(kx) \geq v_p(x) = G\left(\frac{x}{\|p\|^\alpha}\right),$$

if and only if

$$\|T_p\| \leq k^{\frac{1}{\alpha}} \|p\|.$$

□

Theorem 2.4. *Let $T : (V, \|\cdot\|, G, \alpha) \rightarrow (V', \|\cdot\|, G, \alpha)$ be strongly **C**-bounded, and let G be strictly increasing on $[0, 1]$ then T is a bounded linear operator in normed space.*



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Proof. If v_p is strictly increasing for every $p \in V$, then the quasi-inverse v_p^Δ is continuous and $B(p)$ is the unique solution of the equation $x = v_p^\Delta(1 - x)$ i.e.

$$(2.1) \quad B(p) = v_p^\Delta(x)(1 - B(p)).$$

If $v_p(x) = G\left(\frac{x}{\|p\|^\alpha}\right)$, then $v_p^\Delta(x) = \|p\|^\alpha G^\Delta(x)$ and from (2.1) it follows that

$$(2.2) \quad B(p) = \|p\|^\alpha G^\Delta(1 - B(p)).$$

Suppose that T is strongly **C**-bounded, i.e. that

$$(2.3) \quad B(T_p) \leq kB(p), \quad \forall p \in V,$$

where $k \in (0, 1)$.

Then (2.2) and (2.3) imply

$$\|T_p\|^\alpha \leq \frac{B(T_p)}{G^\Delta(1 - B(T_p))} \leq \frac{kB(p)}{G^\Delta(1 - kB(p))} \leq \frac{kB(p)}{G^\Delta(1 - B(p))} = k\|p\|^\alpha.$$

Which means that T is a bounded in normed space. □

The converse of the above theorem is not true, see Example 2.2.

We recall the following theorems from [3].

Theorem 2.5. *Let (V, ν, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$ be PN spaces. A linear map $T : V \rightarrow V'$ is either continuous at every point of V or at no point of V .*

Corollary 2.6. *If $T : (V, \nu, \tau, \tau^*) \rightarrow (V', \mu, \sigma, \sigma^*)$ is linear, then T is continuous if, and only if, it is continuous at θ .*

Theorem 2.7. Every strongly **B**-bounded linear operator T is continuous with respect to the strong topologies in (V, ν, τ, τ^*) and $(V', \mu, \sigma, \sigma^*)$, respectively.

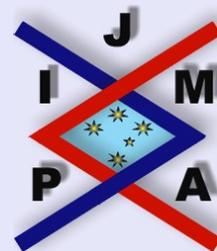
In the following theorem we show that every strongly **C**-bounded linear operator T is continuous.

Theorem 2.8. Every strongly **C**-bounded linear operator T is continuous.

Proof. Due to Corollary 3.1 [3], it suffices to verify that T is continuous at θ . Let $N_{\theta'}(t)$, with $t > 0$, be an arbitrary neighbourhood of θ' . If T is strongly **C**-bounded linear operator then there exist $h \in (0, 1)$ such that for every $t > 0$ and $p \in N_{\theta}(s)$ we note that

$$\mu_{Tp}(t) \geq \nu_p(ht) \geq 1 - ht > 1 - t,$$

so $T_p \in N_{\theta'}(t)$; in other words, T is continuous. □



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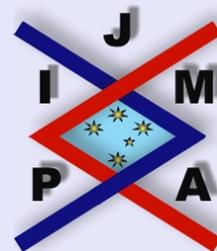
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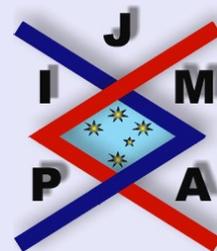
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