
Zbl 1186.76675

Carlen, Eric A.; Carvalho, Maria C.; Le Roux, Jonathan; Loss, Michael; Villani, Cédric

Entropy and chaos in the Kac model. (English)

Kinet. Relat. Models 3, No. 1, 85-122 (2010). ISSN 1937-5093; ISSN 1937-5077
http://dx.doi.org/10.3934/krm.2010.3.85
http://www.aimsciences.org/journals/krm/index.htm

Summary: We investigate the behavior in N of the N–particle entropy functional for Kac's stochastic model of Boltzmann dynamics, and its relation to the entropy function for solutions of Kac's one dimensional nonlinear model Boltzmann equation. We prove results that bring together the notion of propagation of chaos, which Kac introduced in the context of this model, with the problem of estimating the rate of equilibration in the model in entropic terms, showing that the entropic rate of convergence can be arbitrarily slow. Results proved here show that one can in fact use entropy production bounds in Kac's stochastic model to obtain entropic convergence bounds for his non linear model Boltzmann equation, though the problem of obtaining optimal lower bounds of this sort for the original Kac model remains open and the upper bounds obtained here show that this problem is somewhat subtle.

Keywords : entropy; propagation of chaos
Classification :
∗76P05 Molecular or atomic structure
60G50 Sums of independent random variables
54C70 Topological entropy

Zbl pre05770630

Mouhot, C.; Villani, Cédric

Landau damping. (English)

J. Math. Phys. 51, No. 1, Paper No. 015204, 7 p. (2010). ISSN 0022-2488
http://dx.doi.org/10.1063/1.3285283
http://jmp.aip.org/

Classification :
∗82D10 Plasmas

Zbl pre05722367

Villani, Cédric

On the Institut Henri Poincaré. (À propos de l'Institut Henri Poincaré.) (French)

Gaz. Math., Soc. Math. Fr. 124, 91-93 (2010). ISSN 0224-8999

http://smf.emath.fr/en/Publications/Gazette/

See also the author and *A. Comtet* [Eur. Math. Soc. Newsl. 73, 37–39 (2009; Zbl 1189.01070)].

Classification :

∗01A74 History of mathematics at institutions and academies (nonuniversity)

Zbl pre05688245

Loeper, Grégoire; Villani, Cédric

Regularity of optimal transport in curved geometry: the nonfocal case. (English)

Duke Math. J. 151, No. 3, 431-485 (2010). ISSN 0012-7094
http://dx.doi.org/10.1215/00127094-2010-003
http://www.dukemathjournal.org
http://projecteuclid.org/handle/euclid.dmj

The authors study the extension problem of smoothness of optimal transport maps for the quadratic cost function in $\mathbb{R}^n$. They consider only the important case when the cost is the squared geodesic distance on a Riemannian manifold which has many applications in Riemannian geometry. Let M be a smooth connected complete Riemannian manifold and for $(x, y) \in M \times M \setminus \mathrm{cut}(M)$, where $\mathrm{cut}(M)$ denotes the cut locus of M, take coordinate systems (x_i) and (y_j) around x and y, respectively. Set $c(x', y') = d(x', y')^2/2$, where d is the geodesic distance on M, and note that c is C^∞ near (x, y). For $\xi \in T_xM$, $\eta \in T_yM$, define

$$\mathfrak{S}(x, y) \cdot (\xi, \eta) = \tfrac{3}{2} \sum_{i,j,k,l,r,s} (c_{ij,r} c^{r,s} - c_{ij,kl})\, \xi^i \xi^j \eta^k \eta^l,$$

where c_i (resp., $c_{,j}$) denotes the partial derivative with respect to x_i (resp., y_j), evaluated at (x, y), and $c_{i,j}$ is the mixed second derivative with respect to x_i and y_j, and so forth; and $(c^{i,j})$ denotes the inverse of $(c_{i,j})$, always evaluated at (x, y). It is known that this formula defines a covariant tensor, called Ma-Trudinger-Wang tensor (MTW tensor in short). Moreover, if ξ and η are orthogonal unit vectors in T_xM, then $\mathfrak{S}(x, y) \cdot (\xi, \eta)$ coincides with the sectional curvature at x along the plane generated by ξ and η. We assume that

$$\mathfrak{S}(x, y) \geq K|\xi|^2|\eta|^2 \quad \text{whenever} \quad \sum_{i,j} c_{i,j} \xi^i \eta^j = 0,$$

where K is a positive constant (strong MTW condition) or $K = 0$ (weak MTW condition). This condition implies that the sectional curvature of M is bounded below by K. In [Optimal transport. Old and new. Grundlehren 338. Berlin: Springer (2009; Zbl 1156.53003)], *C. Villani* proved a necessary condition for the regularity of optimal transport. In this paper, the authors prove a sufficient condition for regularity under a simplifying nonfocality assumption. For any $\xi \in T_xM$, $|\xi| = 1$, let $t_C(\xi)$ be the first time t such that $(\exp_x(s\xi))_{0 \leq s \leq t'}$ is not minimizing for $t' > t$, and let $t_F(\xi) \geq t_C(\xi)$ be the first time t such that $d_{t\xi}\exp_x$ is not one-to-one. Define $\mathrm{TCL}(x) = \{t\xi \,:\, t = t_C(\xi)\}$ and $\mathrm{TCL}(M) = \cup_x \mathrm{TCL}(x)$. Set also $\mathrm{TFL}(x) = \{t\xi \,:\, t = t_F(\xi)\}$ and $\mathrm{TFL}(M) = \cup_x \mathrm{TFL}(x)$.

We say that the cut locus of M is nonfocal if $\mathrm{TCL}(M) \cap \mathrm{TFL}(M) = \emptyset$. With these assumptions, the authors prove that for any C^∞ positive densities f and g on M, the optimal transport map from $\mu(dx) = f(x)\mathrm{vol}(dx)$ to $\nu(dy) = g(y)\mathrm{vol}(dy)$, with cost function $c = d^2$, is smooth.

Gabjin Yun (Yongin)

Keywords : optimal transport; Ma-Trudinger-Wang tensor; nonfocality; uniform regular manifold

Classification :

∗53C20 Riemannian manifolds (global)
35J60 Nonlinear elliptic equations
49Q20 Variational problems in geometric measure-theoretic setting

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Zbl 1189.01070

Comtet, Alain; **Villani, Cédric**

Institut Henri Poincaré. (English)

Eur. Math. Soc. Newsl. 73, 37-39 (2009). ISSN 1027-488X
http://www.ems-ph.org/journals/journal.php?jrn=news

Classification :

∗01A74 History of mathematics at institutions and academies (nonuniversity)

Zbl 1179.60068

Grunewald, Natalie; **Otto, Felix**; **Villani, Cédric**; **Westdickenberg, Maria G.**

A two-scale approach to logarithmic Sobolev inequalities and the hydrodynamic limit. (English)

Ann. Inst. Henri Poincaré, Probab. Stat. 45, No. 2, 302-351 (2009). ISSN 0246-0203
http://dx.doi.org/10.1214/07-AIHP200
http://www.sciencedirect.com/science/journal/02460203
http://www.imstat.org/aihp/
http://projecteuclid.org/aihp

The authors use the logarithmic Sobolev inequalities (LSI) to deal with the coarse-graining of a lattice system with continuous spin variable, and one provides general conditions in order that a probability measure satisfies a LSI, from there one obtains a criterion for hydrodynamic limit. As an application example, one derives a LSI for a system of spins interacting by Kawasaki dynamics with a Ginzburg-Landau-type potential.

Guy Jumarie (Montréal)

Keywords : logarithmic Sobolev inequality; hydrodynamic limit; Spin system; Hawasaki dynamics; canonical ensembles; coarse0graining

Classification :

∗60K35 Interacting random processes

60J25 Markov processes with continuous parameter
82B21 Continuum models (systems of particles, etc.)

Zbl 1178.53038

Lott, John; Villani, Cedric
Ricci curvature for metric-measure spaces via optimal transport. (English)
Ann. Math. (2) 169, No. 3, 903-991 (2009). ISSN 0003-486X; ISSN 1939-0980
http://dx.doi.org/10.4007/annals.2009.169.903
http://annals.math.princeton.edu/annals/2009/169-3/p04.xhtml
http://annals.math.princeton.edu/annals/about/cover/cover.html
http://pjm.math.berkeley.edu/annals/about/journal/about.html
http://www.jstor.org/journals/0003486X.html

The main results of this very interesting paper, showing that there is a good notion of a compact measured length space having "N-Ricci curvature bounded below by $K \in \mathbb{R}$" for each $N \in [1, +\infty]$ – via optimal transport in the setting of length spaces – which is accompanied with geometric and analytic consequences, have been presented in a previous paper of the first author ["Optimal transport and Ricci curvature for metric-measure spaces", see our review of this paper, that appeared in Surveys in Differential Geometry 11, 229–257 (2007; Zbl 1155.53026)].
The authors provide here complete proofs of these results. In the case of Riemannian manifolds their definitions are equivalent to classical ones. There are many other aspects of the paper, and in particular, six appendices that contain technical and auxiliary results, explanations how to extend the results of the paper from the setting of compact measured length spaces to the setting of complete pointed locally compact measured length spaces, and some bibliographic notes on optimal transport and displacement convexity, which illustrate the authors theory in a convincing way.

Mircea Craioveanu (Timişoara)
Keywords : length space; measured Gromov-Hausdorff convergence; Ricci curvature; optimal transport; Wasserstein space; displacement interpolation; displacement convexity; Wasserstein geodesic; Bishop-Gromov inequality; logarithmic Sobolev inequality; Talagrand inequality; Poincaré inequality; Riemannian manifold; Aleksandrov space
Classification :
*53C23 Global topological methods (a la Gromov)
53C21 Methods of Riemannian geometry (global)
60B05 Probability measures on topological spaces
60B10 Convergence of probability measures
58C35 Integration on manifolds

Zbl 1156.58015

Lu, Peng; Ni, Lei; Vázquez, Juan-Luis; Villani, Cédric
Local Aronson-Bénilan estimates and entropy formulae for porous medium and fast diffusion equations on manifolds. (English)

J. Math. Pures Appl. (9) 91, No. 1, 1-19 (2009). ISSN 0021-7824
http://dx.doi.org/10.1016/j.matpur.2008.09.001
http://www.sciencedirect.com/science/journal/00217824

Summary: We derive local gradient and Laplacian estimates of the Aronson-Bénilan and Li-Yau type for positive solutions of porous medium equations posed on Riemannian manifolds with a lower Ricci curvature bound. We also prove similar results for some fast diffusion equations. Inspired by Perelman's work we discover some new entropy formulae for these equations.

Keywords : porous medium equation; Aronson-Bénilan estimate; Li-Yau type estimate; entropy formula
Classification :
∗58J90 Applications
35Q72 Other PDE from mechanics

Zbl 1156.53003

Villani, Cédric
Optimal transport. Old and new. (English)
Grundlehren der Mathematischen Wissenschaften 338. Berlin: Springer. xxii, 973 p. EUR 99.95/net; SFR 166.00; $ 159.00; £ 79.00 (2009). ISBN 978-3-540-71049-3/hbk
http://dx.doi.org/10.1007/978-3-540-71050-9

As the title suggests, the book is aimed to old and new problems of optimal transport. After the publication of "Topics in optimal transportation" [Graduate Studies in Mathematics 58. (Providence), RI: American Mathematical Society (AMS). (2003; Zbl 1106.90001)], the present book offered to the author a good opportunity to approach differently the whole theory, with alternative proofs and a more probabilistic presentation, and to incorporate new results. Among these new results was John Mather's minimal measures which has a lot to do with optimal transport, and the fact that optimal transport could provide a robust synthetic approach to Ricci curvature bounds. In comparison with the previous book, this approach is oriented more on probability, geometry, and dynamical systems, and less on analysis and physics. According to the author's recommendation, both books can be read independently, or together, and their complementarity can have pedagogical value.
In order to give the reader a feeling for the content and applicability of the results we point out in the sequel the titles of the parts and the chapters: Introduction: (1) Couplings and changes of variables, (2) Three examples of coupling techniques, (3) The founding fathers of optimal transport; Part I – Qualitative description of optimal transport: (4) Basic properties, (5) Cyclical monotonicity and Kantorovich duality, (6) The Wasserstein distances, (7) Displacement interpolation, (8) The Monge-Mather shortening principle, (9) Solution of the Monge problem I – Global approach, (10) Solution of the Monge problem II – Local approach, (11) The Jacobian equation, (12) Smoothness, (13) Qualitative picture; Part II Optimal transport and Riemannian ge-

ometry: (14) Ricci curvature, (15) Otto calculus, (16) Displacement convexity I, (17) Displacement convexity II, (18) Volume control, (19) Density control and local regularity, (20) Infinitesimal displacement convexity, (21) Isoperimetric-type inequalities, (22) Concentration inequalities, (23) Gradient flows I, (24) Gradient flows II – Qualitative properties, (25) Gradient flows III – Functional inequalities; Part III Synthetic treatment of Ricci curvature: (26) Analytic and synthetic points of view, (27) Convergence of metric-measure spaces, (28) Stability of optimal transport, (29) Weak Ricci curvature bounds I – Definition and Stability, (30) Weak Ricci curvature bounds II – Geometric and analytic properties; Conclusions and open problems.

This meticulous work is based on very large bibliography (846 titles) that is converted into a very valuable monograph that presents many statements and theorems written specifically for this approach, complete and self-contained proofs of the most important results, and extensive bibliographical notes. Disseminated throughout the book, several appendices contain either some domains of mathematics useful to non-experts, or proofs of important auxiliary results.

Very useful instruments as List of short statements, List of figures, Index, and Some notable cost functions are accessible at the end of the book.

Mihail Voicu (Iaşi)

Keywords : qualitative description of optimal transport; cyclical monotonicity; Kantorovich duality; Monge-Mather shortening principle; Monge problem; Riemannian geometry; displacement convexity; gradient flows; Ricci curvature; stability of optimal transport; weak Ricci curvature bounds

Classification :

∗53-02 Research monographs (differential geometry)
49-02 Research monographs (calculus of variations)
90-02 Research monographs (optimization)
93-02 Research monographs (systems and control)

Zbl pre05657453

Villani, Cédric

Hypocoercivity. (English)

Mem. Am. Math. Soc. 950, 1-141 (2009). ISSN 0065-9266

Classification :

∗35B40 Asymptotic behavior of solutions of PDE
35K65 Parabolic equations of degenerate type
76P05 Molecular or atomic structure

Zbl pre05640830

Gamba, I.M.; Panferov, V.; Villani, C.

Upper Maxwellian bounds for the spatially homogeneous Boltzmann equation. (English)

Arch. Ration. Mech. Anal. 194, No. 1, 253-282 (2009). ISSN 0003-9527; ISSN 1432-0673

http://dx.doi.org/10.1007/s00205-009-0250-9
http://www.springerlink.com/content/101155/

Summary: For the spatially homogeneous Boltzmann equation with cutoff hard potentials, it is shown that solutions remain bounded from above uniformly in time by a Maxwellian distribution, provided the initial data have a Maxwellian upper bound. The main technique is based on a comparison principle that uses a certain dissipative property of the linear Boltzmann equation. Implications of the technique to propagation of upper Maxwellian bounds in the spatially-inhomogeneous case are discussed.

Classification :
∗76P05 Molecular or atomic structure
82B40 Kinetic theory of gases

Zbl 1172.53022

Figalli, Alessio; Villani, Cédric

An approximation lemma about the cut locus, with applications in optimal transport theory. (English)

Methods Appl. Anal. 15, No. 2, 149-154 (2008). ISSN 1073-2772
http://www.projecteuclid.org/euclid.maa/1234536491
http://www.intlpress.com/MAA/
http://projecteuclid.org/maa

Summary: A path in a Riemannian manifold can be approximated by a path meeting only finitely many times the cut locus of a given point. The proof of this property uses recent works of Itoh-Tanaka and Li-Nirenberg about the differential structure of the cut locus. We present applications in the regularity theory of optimal transport.

Keywords : cut locus; optimal transport; co-area formula
Classification :
∗53C20 Riemannian manifolds (global)
35B65 Smoothness of solutions of PDE
49N99 Miscellaneous problems of optimal control

Zbl 1158.53036

Villani, C.

Stability of a 4th-order curvature condition arising in optimal transport theory. (English)

J. Funct. Anal. 255, No. 9, 2683-2708 (2008). ISSN 0022-1236
http://dx.doi.org/10.1016/j.jfa.2008.07.003
http://www.sciencedirect.com/science/journal/00221236

Summary: A certain curvature condition, introduced by Ma, Trudinger and Wang in

relation with the regularity of optimal transport, is shown to be stable under Gromov-Hausdorff limits, even though the condition implicitly involves fourth order derivatives of the Riemannian metric. Two lines of reasoning are presented with slightly different assumptions, one purely geometric, and another one combining geometry and probability. Then a converse problem is studied: prove some partial regularity for the optimal transport on a perturbation of a Riemannian manifold satisfying a strong form of the Ma-Trudinger-Wang condition.

Keywords : optimal transport; Ma-Trudinger-Wang condition; Gromov-Hausdorff convergence
Classification :
∗53C23 Global topological methods (a la Gromov)
53C21 Methods of Riemannian geometry (global)
49K99 Necessary and sufficient conditions for optimality

Zbl 1156.82313

Villani, Cédric
***H*-theorem and beyond: Boltzmann's entropy in today's mathematics.** (English)
Gallavotti, Giovanni (ed.) et al., Boltzmann's legacy. Papers based on the presentations at the international symposium, Vienna, Austria, June 7–9, 2006. Zürich: European Mathematical Society (EMS). ESI Lectures in Mathematics and Physics, 129-143 (2008). ISBN 978-3-03719-057-9/pbk

Classification :
∗82-03 Historical (statistical mechanics)
01A70 Biographies, obituaries, personalia, bibliographies
01A55 Mathematics in the 19th century
01A60 Mathematics in the 20th century

Zbl 1135.46016

Maggi, F.; Villani, C.
Balls have the worst best Sobolev inequalities. II: Variants and extensions. (English)
Calc. Var. Partial Differ. Equ. 31, No. 1, 47-74 (2008). ISSN 0944-2669; ISSN 1432-0835
http://dx.doi.org/10.1007/s00526-007-0105-x
http://link.springer.de/link/service/journals/00526/

This is the sequel to [J. Geom. Anal. 15, No. 1, 83–121 (2005; Zbl 1086.46021)] about the derivation of sharp Sobolev type inequalities in $\mathbb{R}^n$ by means of mass transport methods. The goal of the present paper is to consider three different kinds of extensions of previous results:
(1) with trace term in Lipschitz domains,

(2) limit inequalities such as trace Faber-Krahn inequalities, trace L^p-logarithmic Sobolev inequalities and trace Moser-Trudinger inequalities,
(3) inequalities on angular domains for which the trace term disappears.

Leszek Skrzypczak (Poznań)
Keywords : Sobolev inequalities; Gagliardo-Nirenberg inequalities; Faber-Krahn inequalities; logarithmic Sobolev inequalities; Moser-Trudinger inequalities; mass transport methods
Classification :
∗46E35 Sobolev spaces and generalizations
26D15 Inequalities for sums, series and integrals of real functions
35R45 Partial differential inequalities

Zbl 1128.76056

Villani, Cédric
Entropy production and convergence to equilibrium. (English)
Golse, François (ed.) et al., Entropy methods for the Boltzmann equation. Lectures from a special semester at the Centre Émil Borel, Institutit H. Poincaré, Paris 2001. Berlin: Springer. Lecture Notes in Mathematics 1916, 1-70 (2008). ISBN 978-3-540-73704-9/pbk
http://dx.doi.org/10.1007/978-3-540-73705-6

Summary: This set of notes was used to complement my short course on the convergence to equilibrium for the Boltzmann equation, given at Institut Henri Poincaré in November–December 2001, as part of the hydrodynamic limits program organized by Stefano Olla and François Golse. The informal style is in accordance with the fact that this is neither a reference book, nor a research paper. The reader can use my review paper, "A review of mathematical topics in collisional kinetic theory" [in: Friedlander, S. (ed.) et al., Handbook of mathematical fluid dynamics. Vol. 1, Amsterdam: Elsevier. 71–305 (2002; Zbl 1170.82369)], as a reference source to dissipate any ambiguity with respect to notation for instance. Apart from minor corrections here and there, the main changes with respect to the original version of the notes were the addition of a final section to present some more recent developments and open directions, and the change of the sign convention for the entropy, to agree with physical tradition.

Classification :
∗76P05 Molecular or atomic structure
82B40 Kinetic theory of gases

Zbl 1125.76001

Rezakhanlou, Fraydoun; Villani, Cédric; Golse, François (ed.); Olla, Stefano (ed.)
Entropy methods for the Boltzmann equation. Lectures from a special semester at the Centre Émil Borel, Institutit H. Poincaré, Paris 2001. (English)

Lecture Notes in Mathematics 1916. Berlin: Springer. xii, 107 p. EUR 29.95/net; SFR 52.50; £ 23.00; $ 44.95 (2008). ISBN 978-3-540-73704-9/pbk
http://dx.doi.org/10.1007/978-3-540-73705-6

The articles of this volume will be reviewed individually.

Classification :

∗76-02 Research monographs (fluid mechanics)
76P05 Molecular or atomic structure
82B40 Kinetic theory of gases

Zbl 1178.35306

Villani, Cédric
Hypocoercive diffusion operators. (English)
Boll. Unione Mat. Ital., Sez. B, Artic. Ric. Mat. (8) 10, No. 2, 257-275 (2007). ISSN 0392-4041

Summary: In many problems coming from mathematical physics, the association of a degenerate diffusion operator with a conservative operator may lead to dissipation in all variables and convergence to equilibrium. One can draw an analogy with the well-studied phenomenon of hypoellipticity in regularity theory, and actually both phenomena have been studied together. Now a distinctive theory of "hypocoercivity" is starting to emerge, with already some striking results, and several challenging open problems. This text (an abbreviated version of the one which I prepared for the International Congress of Mathematicians) will review some of them.

Classification :

∗35Q35 Other equations arising in fluid mechanics
35K55 Nonlinear parabolic equations
82C05 Classical dynamic and nonequilibrium statistical mechanics (general)
82C40 Kinetic theory of gases

Zbl 1125.53026

Figalli, A.; Villani, C.
Strong displacement convexity on Riemannian manifolds. (English)
Math. Z. 257, No. 2, 251-259 (2007). ISSN 0025-5874; ISSN 1432-1823
http://dx.doi.org/10.1007/s00209-007-0124-5
http://link.springer.de/link/service/journals/00209/

Ricci curvature bounds in Riemannian geometry are known to be equivalent to the weak convexity of certain functionals in the space of probability measures. The authors show that the weak convexity can be reinforced into strong convexity, thus solving a question left open in [Lott and Villani, Ann. Math., to appear].

In this paper, the authors first give a definition: Let ν be a reference measure on an n-dimensional Riemannian manifold (M, g), absolutely continuous with respect to the volume measure. Let $U : \mathbb{R}_+ \to \mathbb{R}$ be a continuous convex function with $U(0) = 0$; let $U'(\infty)$ be the limit of $U(r)/r$ as $r \to \infty$. Let μ be a probability measure on M and let $\mu = \rho\nu + \mu_s$ be its Lebesgue decomposition with respect to ν.
(i) If $U(\rho)$ is bounded below by a ν-integrable function, then the quantity $U_\nu(\mu)$ is defined by the formula

$$U_\nu(\mu) = \int_M U(\rho(x))\nu(dx) + U'(\infty)\mu_s[M].$$

(ii) If π is a probability measure on $M \times M$, admitting μ as first martingale, β is a positive function on $M \times M$, and $\beta U(\rho/\beta)$ (as a function of x, y) is bounded below by a ν-integrable function of x, then the quantity $U^\beta_{\pi,\nu}(\mu)$ is defined by the formula

$$U^\beta_{\pi,\nu}(\mu) = \int_{M\times M} U\left(\frac{\rho(x)}{\beta(x,y)}\right)\beta(x,y)\pi(dy|x)\nu(dx) + U'(\infty)\mu_s[M],$$

where $\pi(dy|x)$ is the disintegration of $\pi(dxdy)$ with respect to the x variable.
Then they obtain the following main result: Theorem. Let U, ν and β be as in the above definition. Assume that U is Lipschitz. For each $a > 0$, define $U_a(r) = U(ar)/a$. Then (i) If $(U_a)_\nu$ is weakly λ-a.c.c.s. displacement convex for any $a \in (0, 1]$, U_ν is λ-displacement convex; (ii) If $(U_a)_\nu$ is weakly λ-a.c.c.s. displacement convex with distortion β for any $a \in (0, 1]$, U_ν is displacement convex with distortion β.
From this theorem, they obtain the following Corollary. Let M be a smooth complete Riemannian manifold with nonnegative Ricci curvature and dimension n. Let $U(r) = -r^{1-\frac{1}{n}}$, and let ν be the volume measure on M. Then U_ν is displacement convex on the set of probability measures $P(M)$, where $p = 2$ if $n \geq 3$, and p is any real number greater than 2 if $n = 2$.

Huafei Sun (Beijing)

Keywords : weak convexity; strong displacement convexity; Ricci curvature
Classification :
∗53C20 Riemannian manifolds (global)

Zbl 1119.53028

Lott, John; Villani, Cédric

Weak curvature conditions and functional inequalities. (English)

J. Funct. Anal. 245, No. 1, 311-333 (2007). ISSN 0022-1236
http://dx.doi.org/10.1016/j.jfa.2006.10.018
http://www.sciencedirect.com/science/journal/00221236

In the paper the authors give sufficient conditions for a measured length space (X, d, ν) to admit local and global Poincare inequalities. lt is a continuation of the paper of authors entitled Ricci curvature for metric-measure spaces via optimal transport which will appear in Ann. Math.. Following on the work of Cheeger and Colding they introduce a certain condition DM on a measured length space. Roughly speaking the condition

DM says that there is a dynamical democratic transference plan so that a given point is not hit too often by the geodesics. Then they show that this condition together with a doubling condition on ν, implies a scale invariant local Poincaré inequality. They show that if (X, d, ν) has nonnegative N-Ricci curvature and has unique minimizing geodesics between almost all pairs of points then it satisfies DM, with constant 2^N. The condition DM is preserved by measured Gromov-Hausdorff limits. Based on it they prove a Sobolev inequality for measured length spaces with N-Ricci curvature bounded below by $K > 0$. Finally the authors derive a sharp global Poincaré inequality.

Andrzej Szczepański (Gdańsk)

Keywords : Poincaré inequalities; Ricci curvature; metric-measure spaces; Sobolev inequality; Gromov-Hausdorff limits

Classification :

∗53C23 Global topological methods (a la Gromov)
46E35 Sobolev spaces and generalizations
31C15 Generalizations of potentials, etc.
49Q15 Geometric measure and integration theory, etc.
53C21 Methods of Riemannian geometry (global)
58C35 Integration on manifolds

Zbl 1113.60093

Bolley, François; Guillin, Arnaud; Villani, Cédric

Quantitative concentration inequalities for empirical measures on non-compact spaces. (English)

Probab. Theory Relat. Fields 137, No. 3-4, 541-593 (2007). ISSN 0178-8051; ISSN 1432-2064
http://dx.doi.org/10.1007/s00440-006-0004-7
http://link.springer.de/link/service/journals/00440/

Quantitative concentration estimates for the empirical measure of many independent variables, in transportation distances, are established. Typical estimates are of the form

$$P\left[\sup_{\|\varphi\|_{\mathrm{Lip}}\le 1}\left(N^{-1}\sum_{i=1}^{N}\varphi(X_t^i)-\int\varphi\,d\mu_t\right)>\varepsilon\right]\le Ce^{-\lambda N\varepsilon^2},$$

where $\|\varphi\|_{\mathrm{Lip}} = \sup_{x\neq y}|\varphi(x)-\varphi(y)|d^{-1}(x,y)$ and $d(\cdot)$ stands for the distance in phase space (say Euclidean norm in $\mathbb{R}^d$). The core of estimates is based on variants of Sanov's theorem.

These results are applied to provide some error bounds for particle simulations in a simple mean-field kinetic model for granular media. A system of interacting particles is governed by the system of coupled stochastic differential equations. The tools include coupling arguments, as well as regularity and moment estimates for solutions of differential equations.

Alex V. Kolnogorov (Novgorod)

Keywords : transport inequalities; Sanov Theorem

Classification :

∗60K35 Interacting random processes
60H15 Stochastic partial differential equations

Zbl pre05200851

Lott, John; Villani, Cédric

Hamilton-Jacobi semigroup on length spaces and applications. (English)

J. Math. Pures Appl. (9) 88, No. 3, 219-229 (2007). ISSN 0021-7824
http://dx.doi.org/10.1016/j.matpur.2007.06.003
http://www.sciencedirect.com/science/journal/00217824

Summary: We define a Hamilton-Jacobi semigroup acting on continuous functions on a compact length space. Following a strategy of Bobkov, Gentil and Ledoux, we use some basic properties of the semigroup to study geometric inequalities related to concentration of measure. Our main results are that (1) a Talagrand inequality on a measured length space implies a global Poincaré inequality and (2) if the space satisfies a doubling condition, a local Poincaré inequality and a log-Sobolev inequality then it also satisfies a Talagrand inequality.

Keywords : Metric-measure spaces; Hamilton-Jacobi semigroup; Talagrand inequality; logarithmic Sobolev inequality; Poincaré inequality; Ricci curvature

Classification :

∗35-99 Partial differential equations (PDE)
37-99 Dynamic systems and ergodic theory
47-99 Operator theory

Zbl 1184.82002

Villani, Cédric

Introduction for Carlo's special issue. (English)

J. Stat. Phys. 124, No. 2-4, 271-273 (2006). ISSN 0022-4715; ISSN 1572-9613
http://www.springerlink.com/openurl.asp?genre=journalissn=0022-4715

Classification :

∗82-03 Historical (statistical mechanics)
01A70 Biographies, obituaries, personalia, bibliographies

Zbl 1141.53030

Villani, Cédric

Optimal transport and Ricci curvature. (Transport optimal et courbure de Ricci.) (French)

Sémin. Équ. Dériv. Partielles, Éc. Polytech., Cent. Math., Palaiseau 2005-2006, VII-1-VII18 (2006).

The author presents some connections between the Monge-Kantorovich optimal transport and certain problems of Riemannian geometry. Especially, links with the Ricci curvature are analyzed. Also, the paper contains a list of some open problems.

Adrian Sandovici (Piatra Neamt)

Keywords : optimal transport; Riemannian geometry; Ricci curvature
Classification :
∗53C20 Riemannian manifolds (global)
49Q20 Variational problems in geometric measure-theoretic setting
35J60 Nonlinear elliptic equations

Zbl 1134.82040

Villani, Cédric

Mathematics of granular materials. (English)

J. Stat. Phys. 124, No. 2-4, 781-822 (2006). ISSN 0022-4715; ISSN 1572-9613
http://dx.doi.org/10.1007/s10955-006-9038-6
http://www.springerlink.com/openurl.asp?genre=journalissn=0022-4715

Summary: This is a short and somewhat informal review on the most mathematical parts of the kinetic theory of granular media, intended for physicists and for mathematicians outside the field.

Keywords : granular materials; Boltzmann equation; inelastic collisions; homogeneous cooling states
Classification :
∗82C40 Kinetic theory of gases
76T25 Granular flows
82-02 Research monographs (statistical mechanics)

Zbl 1130.35027

Villani, Cédric

Hypocoercive diffusion operators. (English)

Sanz-Solé, Marta (ed.) et al., Proceedings of the international congress of mathematicians (ICM), Madrid, Spain, August 22–30, 2006. Volume III: Invited lectures. Zürich: European Mathematical Society (EMS). 473-498 (2006). ISBN 978-3-03719-022-7/hbk

Summary: In many problems coming from mathematical physics, the association of a degenerate diffusion operator with a conservative operator may lead to dissipation in all variables and convergence to equilibrium. One can draw an analogy with the well-studied phenomenon of hypoellipticity in regularity theory, and actually both phenomena have been studied together. Now a distinctive theory of "hypocoercivity" is starting to emerge, with already some striking results, and several challenging open problems.

Keywords : Hypocoercivity, hypoellipticity, diffusion equations, spectral gap, logarithmic Sobolev inequalities, Fokker-Planck and Boltzmann equations, H Theorem
Classification :
*35H10 Hypoelliptic equations
35B40 Asymptotic behavior of solutions of PDE
35K70 Ultraparabolic (etc.) problems
76P05 Molecular or atomic structure
82C40 Kinetic theory of gases

Zbl 1082.76105

Carrillo, José A.; McCann, Robert J.; Villani, Cédric

Contractions in the 2-Wasserstein length space and thermalization of granular media. (English)

Arch. Ration. Mech. Anal. 179, No. 2, 217-263 (2006). ISSN 0003-9527; ISSN 1432-0673
http://dx.doi.org/10.1007/s00205-005-0386-1
http://www.springerlink.com/content/101155/

Summary: An algebraic decay rate is derived which bounds the time required for velocities to equilibrate in a spatially homogeneous flow-through model representing the continuum limit of a gas of particles interacting through slightly inelastic collisions. This rate is obtained by reformulating the dynamical problem as the gradient flow of a convex energy on an infinite-dimensional manifold. An abstract theory is developed for gradient flows in length spaces, which shows how degenerate convexity (or even non-convexity) – if uniformly controlled – quantify contractivity (limit expansivity) of the flow.

Keywords : convex energy; infinite-dimensional manifold; degenerate convexity
Classification :
*76T25 Granular flows
74E20 Granularity

Zbl 1162.82316

Desvillettes, L.; Villani, C.

On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. (English)

Invent. Math. 159, No. 2, 245-316 (2005). ISSN 0020-9910; ISSN 1432-1297
http://dx.doi.org/10.1007/s00222-004-0389-9
http://link.springer.de/link/service/journals/00222/

Summary: As part of our study of convergence to equilibrium for spatially inhomogeneous kinetic equations, started in [Commun. Pure Appl. Math. 54, No. 1, 1–42 (2001; Zbl 1029.82032)], we derive estimates on the rate of convergence to equilibrium

for solutions of the Boltzmann equation, like $O(t^{-\infty})$. Our results hold conditionally to some strong but natural estimates of smoothness, decay at large velocities and strict positivity, which at the moment have only been established in certain particular cases. Among the most important steps in our proof are:
1) quantitative variants of Boltzmann's H-theorem, based on symmetry features, hypercontractivity and information-theoretical tools;
2) a new, quantitative version of the instability of the hydrodynamic description for non-small Knudsen number;
3) some functional inequalities with geometrical content, in particular the Korn-type inequality; and
4) the study of a system of coupled differential inequalities of second order.
We also briefly point out the particular role of conformal velocity fields, when they are allowed by the geometry of the problem.

Classification :
∗82C40 Kinetic theory of gases
35B40 Asymptotic behavior of solutions of PDE
35F20 General theory of first order nonlinear PDE

Zbl 1087.60008

Bolley, François; **Villani, Cédric**
Weighted Csiszár-Kullback-Pinsker inequalities and applications to transportation inequalities. (English)
Ann. Fac. Sci. Toulouse, VI. Sér., Math. 14, No. 3, 331-352 (2005). ISSN 0240-2963
numdam:AFST$_2$005$_{61}$4$_{33}$31$_0$
http : //afst.cedram.org/cgi − bin/browse
http : //www.numdam.org/numdam − bin/feuilleter?j = AFSTsl = 1

The Csiszár-Kullback-Pinsker inequality compares the total variation of the difference of two probability measures and the Kullback information of one measure with respect to another one. The authors strengthen this inequality by allowing weights in the total variation norm. The class of admissible weights depends on the decay properties of the reference probability measure. This result is used to derive transportation inequalities involving Wasserstein distances for various exponents. An application to transportation inequalities for random dynamical systems is given.

Anatoly N. Kochubei (Kyïv)
Keywords : Csiszár-Kullback-Pinsker inequality; Wasserstein distance; Kullback information; transportation inequalities; random dynamical system
Classification :
∗60B05 Probability measures on topological spaces
60A10 Probabilistic measure theory
28A33 Spaces of measures
37H10 Random and stochastic difference and differential equations

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Zbl 1086.46021

Maggi, Francesco; Villani, Cédric

Balls have the worst best Sobolev inequalities. (English)

J. Geom. Anal. 15, No. 1, 83-121 (2005). ISSN 1050-6926; ISSN 1559-002X
http://www.springerlink.com/content/120876/
http://www.jga.wustl.edu/contents/content.html

The authors deal with Sobolev inequalities with trace term. These are inequalities of the type:

$$\|f\|_{L^{p^*}(\Omega)} \leq S_n(p)\|f\|_{L^p(\Omega)} + T_n(p)\|f\|_{L^{p^\sharp}(\partial\Omega)},$$
$$\|f\|^p_{L^{p^*}(\Omega)} \leq S^p_n(p)\|f\|_{L^p(\Omega)} + C^p_n(p)\|f\|^p_{L^{p^\sharp}(\partial\Omega)},$$

where Ω is a locally Lipschitz open domain in $\mathbb{R}^n$, $p^* = \frac{np}{n-p}$, $p^\sharp = \frac{(n-1)p}{n-p}$, and $S_n(p)$ is the optimal constant in the Sobolev classical inequality. Inequalities of this type where first studied by H. Brézis and E. Lieb. Sharp versions of the inequalities were proved by the authors. The isoperimetric inequality and the sharp Euclidean Sobolev embedding are particular cases. The main argument used in the proofs is an optimal mass transportation argument.

Leszek Skrzypczak (Poznań)

Keywords : Sobolev inequality; traces; isoperimetric inequality; mass transportation
Classification :
∗46E35 Sobolev spaces and generalizations
26D15 Inequalities for sums, series and integrals of real functions

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Zbl pre05130613

Villani, Cédric

Entropy production and convergence to equilibrium for the Boltzmann equation. (English)

Zambrini, Jean-Claude (ed.), XIVth international congress on mathematical physics (ICMP 2003), Lisbon, Portugal, 28 July – 2 August 2003. Selected papers based on the presentation at the conference. Hackensack, NJ: World Scientific. 130-144 (2005). ISBN 981-256-201-X/hbk

Classification :
∗82C40 Kinetic theory of gases
76P05 Molecular or atomic structure
82-02 Research monographs (statistical mechanics)
82B40 Kinetic theory of gases
82D05 Gases

Zbl 1134.35375

Villani, Cédric

Trend to equilibrium for dissipative equations, functional inequalities and mass transportation. (English)

Carvalho, M. C. (ed.) et al., Recent advances in the theory and applications of mass transport. Papers from the summer school on mass transportation methods in kinetic theory and hydrodynamics, Ponta Delgada, Azores, Portugal, September 4–9, 2000. Providence, RI: American Mathematical Society (AMS). Contemporary Mathematics 353, 95-109 (2004). ISBN 0-8218-3278-6/pbk

From the introduction: In this review paper, we explain how the trend to equilibrium for some dissipative equations can be studied via certain classes of functional inequalities, with the tool of mass transportation directly or indirectly involved. This short account is mainly intended to give an idea of the subject. A more precise review is given in Chapter 9 of [C. Villani, Topics in optimal transportation, Amer. Math. Soc., Providence, RI (2003; Zbl 1106.90001)], or in the research papers [F. Otto and C. Villani, J. Funct. Anal. 173, No. 2, 361–400 (2000; Zbl 0985.58019); J. Math. Pures Appl. (9) 80, No. 7, 697–700 (2001; Zbl 1134.35312)]. We also discuss a few directions of possible future research.

Olaf Ninnemann (Berlin)

Classification :

∗35L55 Higher order hyperbolic systems
35K57 Reaction-diffusion equations
49Q20 Variational problems in geometric measure-theoretic setting
60E15 Inequalities in probability theory
82C40 Kinetic theory of gases

Zbl 1106.82031

Gamba, I.M.; Panferov, V.; Villani, C.

On the Boltzmann equation for diffusively excited granular media. (English) Commun. Math. Phys. 246, No. 3, 503-541 (2004). ISSN 0010-3616; ISSN 1432-0916

http://dx.doi.org/10.1007/s00220-003-1051-5

http://link.springer.de/link/service/journals/00220/

http://projecteuclid.org/DPubS?service=UIversion=1.0verb=Displaypage=pasthandle=euclid.cm

Summary: We study the Boltzmann equation for a space-homogeneous gas of inelastic hard spheres, with a diffusive term representing a random background forcing

$$\partial_t f - \mu \Delta_v f = Q(f, f), \quad v \in \mathbb{R}^N, t > 0.$$

Under the assumption that the initial datum f_0 is a nonnegative $L^2(N)$ function, with bounded mass and kinetic energy (second moment), we prove the existence of a solution to this model, which instantaneously becomes smooth and rapidly decaying. Under a weak additional assumption of bounded third moment, the solution is shown to be unique. We also establish the existence (but not uniqueness) of a stationary solution. In addition we show that the high-velocity tails of both the stationary and time-dependent particle distribution functions are overpopulated with respect to the Maxwellian distribution, as conjectured by previous authors, and we prove pointwise lower estimates for

the solutions.

Classification :

∗82C40 Kinetic theory of gases
76P05 Molecular or atomic structure
76T25 Granular flows
35F20 General theory of first order nonlinear PDE

Zbl 1069.35011

Desvillettes, L.; Villani, C.

Rate of convergence toward the equilibrium in degenerate settings. (English)

Monaco, Roberto (ed.) et al., "WASCOM 2003" – Proceedings ot the 12th conference on waves and stability in continuous media, Villasimius, Italy, June 1–7, 2003. River Edge, NJ: World Scientific. 153-165 (2004). ISBN 981-238-748-X/hbk

Summary: We describe here some of the aspects of a general method enabling to yield quantitative estimates about the rate of convergence toward the equilibrium. We are interested in solutions of PDEs in which dissipative effects can be easily obtained only with respect to a part of the variables. Then, one has to use in a subtle way the interplay between the different variables in order to conclude. The example of spatially inhomogeneous kinetic equations (in a confining potential, or in a box with boundary conditions) is detailed.

Keywords : Boltzmann equation; spatially inhomogeneous kinetic equations

Classification :

∗35B40 Asymptotic behavior of solutions of PDE

Zbl 1063.76086

Mouhot, Clément; Villani, Cédric

Regularity theory for the spatially homogeneous Boltzmann equation with cut-off. (English)

Arch. Ration. Mech. Anal. 173, No. 2, 169-212 (2004). ISSN 0003-9527; ISSN 1432-0673
http://dx.doi.org/10.1007/s00205-004-0316-7
http://www.springerlink.com/content/101155/

Summary: We develop a regularity theory for spatially homogeneous Boltzmann equation with cut-off and hard potentials (for instance, hard spheres), by (i) revisiting the L^p theory to obtain constructive bounds; (ii) establishing propagation of smoothness and singularities, (iii) obtaining estimates on the decay of singularities of the initial data. Our proofs are based on a detailed study of the "regularity of the gain operator". An application to the long-time behavior is presented.

Keywords : singularities; gain operator; long-time behavior

Classification :

∗76P05 Molecular or atomic structure
45K05 Integro-partial differential equations

Zbl 1063.35109

Arnold, A.; Carrillo, J.A.; Desvillettes, L.; Dolbeault, J.; Jüngel, A.; Lederman, C.; Markowich, P.A.; Toscani, G.; Villani, C.

Entropies and equilibria of many-particle systems: an essay on recent research. (English)

Monatsh. Math. 142, No. 1-2, 35-43 (2004). ISSN 0026-9255; ISSN 1436-5081
http://dx.doi.org/10.1007/s00605-004-0239-2
http://link.springer.de/link/service/journals/00605/

In this essay the authors briefly review the main ideas developed in the last years concerning the entropy production method for kinetic equations of various types and the relevance and expected impact of such results. These results were a major breakthrough in analyzing in particular the rate of convergence to equilibrium of solutions to this type of equation establishing the "almost" truth of Cercignani's conjecture.

Manuel Portilheiro (Coimbra)

Keywords : entropy production method; kinetic equations; large-time behavior; rate of convergence, Cercignani's conjecture

Classification :

∗35L60 First-order nonlinear hyperbolic equations
76P05 Molecular or atomic structure
82B40 Kinetic theory of gases
35-02 Research monographs (partial differential equations)

Zbl 1048.26010

Cordero-Erausquin, D.; Nazaret, B.; Villani, C.

A mass-transportation approach to sharp Sobolev and Gagliardo-Nirenberg inequalities. (English)

Adv. Math. 182, No. 2, 307-332 (2004). ISSN 0001-8708
http://dx.doi.org/10.1016/S0001-8708(03)00080-X
http://www.sciencedirect.com/science/journal/00018708

The authors give a new proof of the sharp Sobolev embedding in R^n,

$$\left(\int_{R^n} |f(x)|^{p^*}\,dx\right)^{1/p^*} \le S_n(p)\,\left(\int_{R^n} |\nabla f|^p\,dx\right)^{1/p}$$

where

$$1 < p < n, \quad 1 - n/p = -n/p^*.$$

The sharp constants $S_n(p)$ and related extremal functions are calculated using new measure-theoretical arguments, called mass-transportation. Similar results are obtained for $p = 1$ (related to isoperimetrical problems). Euclidean norms are replaced by more general norms. Some results for related Gagliardo-Nirenberg inequalities are obtained.

Hans Triebel (Jena)

Keywords : Sobolev inequalities; Gagliardo-Nirenberg inequalities; mass-transportation
Classification :
∗26D10 Inequalities involving derivatives, diff. and integral operators
46E35 Sobolev spaces and generalizations
49Q20 Variational problems in geometric measure-theoretic setting

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Zbl 1044.83007

Alexandre, R.; Villani, C.

On the Landau approximation in plasma physics. (English)

Ann. Inst. Henri Poincaré, Anal. Non Linéaire 21, No. 1, 61-95 (2004). ISSN 0294-1449
http://dx.doi.org/10.1016/j.anihpc.2002.12.001
numdam:AIHPC$_2$004$_{2\,1_{16}\,1_0}$
*http : //www.sciencedirect.com/science/journal/*02941449

Summary: This paper studies the approximation of the Boltzmann equation by the Landau equation in a regime when grazing collisions prevail. While all previous results in the subject were limited to the spatially homogeneous case, here we manage to cover the general, space-dependent situation, assuming only basic physical estimates of finite mass, energy, entropy and entropy production. The proofs are based on the recent results and methods introduced previously in [Commun. Pure Appl. Math. 55, 30–70 (2002; Zbl 1029.82036)] by both authors, and the entropy production smoothing effects established in [*R. Alexandre* et al., Arch. Ration. Mech. Anal. 152, 327–355 (2000; Zbl 0968.76076)]. We are able to treat realistic singularities of Coulomb type, and approximations of the Debye cut. However, our method only works for finite-time intervals, while the Landau equation is supposed to describe long-time corrections to the Vlasov-Poisson equation. If the mean-field interaction is neglected, then our results apply to physically relevant situations after a time rescaling.

Keywords : Boltzmann equation; Landau equation; singularities
Classification :
∗82D10 Plasmas
35A35 Theoretical approximation to solutions of PDE
35F20 General theory of first order nonlinear PDE
45K05 Integro-partial differential equations
76Y05 Nonclassical hydrodynamics
76P05 Molecular or atomic structure

Zbl 1106.90001

Villani, Cédric

Topics in optimal transportation. (English)

Graduate Studies in Mathematics 58. Providence, RI: American Mathematical Society (AMS). xvi, 370 p. $ 59.00 (2003). ISBN 0-8218-3312-X/hbk

This is a very interesting book: it is the first comprehensive introduction to the theory of mass transportation with its many – and sometimes unexpected – applications. In a novel approach to the subject, the book both surveys the topic and includes a chapter of problems, making it a particularly useful graduate textbook.

In 1781, Gaspard Monge defined the problem of "optimal transportation" (or the transferring of mass with the least possible amount of work), with applications to engineering in mind. In 1942, Leonid Kantorovich applied the newborn machinery of linear programming to Monge's problem, with applications to economics in mind. In 1987, Yann Brenier used optimal transportation to prove a new projection theorem on the set of measure preserving maps, with applications to fluid mechanics in mind.

Each of these contributions marked the beginning of a whole mathematical theory, with many unexpected ramifications. Nowadays, the Monge-Kantorovich problem is used and studied by researchers from extremely diverse horizons, including probability theory, functional analysis, isoperimetry, partial differential equations, and even meteorology.

These notes are definitely not intended to be exhaustive, and should rather be seen as an introduction to the subject. Their reading can be complemented by some of the reference texts which have appeared recently. In particular, mention should be made of the two-volume work by *S. T. Rachev* and *L. Rüschendorf* [Mass transportation problems. Vols. I, II. Springer, New York (1998; Zbl 0990.60500)], which depicts many applications of Monge-Kantorovich distances to various problems, together with the classical theory of the optimal transportation problem in a very abstract setting; the survey of Evans, which can be considered as an introduction to the subject and describes several applications of the L^1 theory (i.e., when the cost function is a distance) which are not covered in these notes; the extremely clear lecture notes by *L. Ambrosio* [Lect. Notes Math. 1812, 1–52 (2003; Zbl 1047.35001)], centred on the L^1 theory from the point of view of calculus of variations; and also the lecture notes by *J. Urbas* ["Mass transfer problems", Lecture notes, Univ. Bonn, 1997–1998; per bibl.], which are a marvelous reference for the regularity theory of the Monge-Ampère equation arising in mass transportation. Also recommended is a very pedagogical and rather complete article recently written by *L. Ambrosio* and *A. Pratelli* [Lect. Notes Math. 1813, 123–160 (2003; Zbl 1065.49026)], focused on the L^1 theory, from which many remarks and examples are extracted here.

The present volume does not go too deeply into some of the aspects which are very well treated in the above-mentioned references: in particular, the L^1 theory is just sketched, and so is the regularity theory developed by Caffarelli and by Urbas. Several topics are hardly mentioned, or not at all: the application of mass transportation to the problem of shape optimization, as developed by Bouchitté and Buttazzo; the fascinating semi-geostrophic system in meteorology, whose links with optimal transportation are now understood thanks to the amazing work of Cullen, Purser and collaborators; or appli-

cations to image processing, developed by Tannenbaum and his group. On the other hand, this text is a good elementary reference source for such topics as displacement interpolation and its applications to functional inequalities with a geometrical content, or the differential viewpoint of Otto, which has proven so successful in various contexts (like the study of rates of equilibration for certain dissipative equations). The proofs are kept as simple as possible throughout the book, are understandable by non-expert students. Many results without proofs, either to convey a better intuition, or to give an account of recent research in the field.

Originating from a graduate course, the present volume is intended for graduate students and researchers, covering both theory and applications. Readers are only assumed to be familiar with the basics of measure theory and functional analysis. (Mainly cited from the publisher notes)

Olaf Ninnemann (Berlin)

Classification :

∗90-02 Research monographs (optimization)
28D05 Measure-preserving transformations
35B65 Smoothness of solutions of PDE
35J60 Nonlinear elliptic equations
49N90 Applications of optimal control and differential games
49Q20 Variational problems in geometric measure-theoretic setting
90B20 Highway traffic

Zbl 1106.65324

Pareschi, L.; Toscani, G.; Villani, C.

Spectral methods for the non cut-off Boltzmann equation and numerical grazing collision limit. (English)

Numer. Math. 93, No. 3, 527-548 (2003). ISSN 0029-599X; ISSN 0945-3245
http://dx.doi.org/10.1007/s002110100384
http://link.springer.de/link/service/journals/00211/

Summary: In this paper we study the numerical passage from the spatially homogeneous Boltzmann equation without cut-off to the Fokker-Planck-Landau equation in the so-called grazing collision limit. To this aim we derive a Fourier spectral method for the non cut-off Boltzmann equation in the spirit of *L. Pareschi* and *B. Perthame* [Transp. Theory Stat. Phys. 25, No. 3-5, 369–382 (1996; Zbl 0870.76074)] and *L. Pareschi* and *G. Russo* [SIAM J. Numer. Anal. 37, No. 4, 1217–1245 (2000; Zbl 1049.76055)]. We show that the kernel modes that define the spectral method have the correct grazing collision limit providing a consistent spectral method for the limiting Fokker-Planck-Landau equation. In particular, for small values of the scattering angle, we derive an approximate formula for the kernel modes of the non cut-off Boltzmann equation which, similarly to the Fokker-Planck-Landau case, can be computed with a fast algorithm. The uniform spectral accuracy of the method with respect to the grazing collision parameter is also proved.

Classification :

∗65M70 Spectral, collocation and related methods (IVP of PDE)
76M22 Spectral methods
76P05 Molecular or atomic structure
82C40 Kinetic theory of gases

Zbl 1073.35127

Carrillo, José A.; McCann, Robert J.; Villani, Cédric

Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates. (English)

Rev. Mat. Iberoam. 19, No. 3, 971-1018 (2003). ISSN 0213-2230
http://projecteuclid.org/rmi
http://www.uam.es/departamentos/ciencias/matematicas/ibero/irevista.htm

The authors study asymptotic behaviour of the equation

$$\frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \nabla (U'(\rho) + V + W * \rho)) \,,$$

where the unknown ρ is a probability measure on $\mathbb{R}^d$, $d \geq 1$, U is the density of internal energy, V is a confinement potential and W is an interaction potential. The functional

$$F(\rho) = \int_{\mathbb{R}^d} U(\rho)\, dx + \int_{\mathbb{R}^d} V(x)\, d\rho(x) + \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} W(x-y)\, d\rho(x) d\rho(y)$$

is the entropy, or free energy, associated with the equation (1). In many cases the competition between the potential determines a unique minimizer ρ_∞ for F. The authors find conditions on U, V, W ensuring that this is true and proceed to determine whether the solutions to (1) converge to ρ_∞, and how fast. The rate of convergence is measured in terms of the relative free energy $F(\rho|\rho_\infty) = F(\rho) - F(\rho_\infty)$. The treatment of the problem is fairly comprehensive and, following the papers, can be summarized in the heuristic rules:
Rule 1: A uniformly convex confinement potential implies an exponential decay to equilibrium. Moreover, if the convexity of the confinement potential is strong enough, it can overcome a lack of convexity of the interaction potential. Rule 2: When the center of mass is fixed, then a uniformly convex interaction potential implies an exponential decay to equilibrium.
Rule 3: When the interaction potential in only degenerately convex and there is no diffusion, then the decay is in general only algebraic.
Rule 4: In presence of (linear or superlinear) diffusion, a degenerately convex interaction potential induces an exponential trend to equilibrium.
Rule 5: In presence of diffusion, the interaction potential is able to drive the system to equilibrium even if the center of mass is moving.
The proofs use the energy dissipation method which, due to the structure of (1), can be applied here via the theory of logarithmic Sobolev inequalities. The authors use two approaches for proving the necessary inequalities: one is based on Bakry and Emery method [*D. Bakry* and *M. Emery*, In: Sémin. de probabilités XIX, Univ. Strasbourg

1983/84, Proc., Lect. Notes Math. 1123, 177–206 (1985; Zbl 0561.60080)] and the other uses the so called HWI method, which involves interpolation inequalities relating the entropy functional, Wasserstein metric and Fisher information, introduced by Otto and Villani [*F. Otto* and *C. Villani*, J. Funct. Anal., 173, 361–400 (2000; Zbl 0985.58019)]. The authors point out that the HWI approach, when applicable, is preferable to the Bakry-Emery argument as it is more direct and does not require a priori knowledge that $\rho \to \rho_\infty$. However, the latter does not use explicitly mass transportation and allows to avoid difficulties related to smoothness issues associated with it.

Jacek Banasiak (Durban)

Keywords : convergence to equilibrium; generalized log-Sobolev inequalities; Wasserstein distance; granular media equation; Bakry and Emery method; HWI method

Classification :

∗35K65 Parabolic equations of degenerate type

35K55 Nonlinear parabolic equations

35B40 Asymptotic behavior of solutions of PDE

Zbl 1041.82018

Villani, Cédric

Cercignani's conjecture is sometimes true and always almost true. (English)

Commun. Math. Phys. 234, No. 3, 455-490 (2003). ISSN 0010-3616; ISSN 1432-0916

http://dx.doi.org/10.1007/s00220-002-0777-1

http://link.springer.de/link/service/journals/00220/

http://projecteuclid.org/DPubS?service=UIversion=1.0verb=Displaypage=pasthandle=euclid.cm

Summary: We establish several new functional inequalities comparing Boltzmann's entropy production functional with the relative H functional. First we prove a longstanding conjecture by Cercignani under the nonphysical assumption that the Boltzmann collision kernel is superquadratic at infinity. The proof rests on the method introduced in [*G. Toscani* and *C. Villani*, Commun. Math. Phys. 203, 667–706 (1999; Zbl 0944.35066)] combined with a novel use of the Blachman-Stam inequality. If the superquadraticity assumption is not satisfied, then it is known that Cercignani's conjecture is not true; however we establish a slightly weakened version of it for all physically relevant collision kernels, thus extending previous results from the paper cited above. Finally, we consider the entropy-entropy production version of Kac's spectral gap problem and obtain estimates about the dependence of the constants with respect to the dimension. The first two results are sharp in some sense, and the third one is likely to be, too; they contain all previously known entropy estimates as particular cases. This gives a first coherent picture of the study of entropy production, according to a program started by *E. Carlen* and *M. Carvalho* [J. Stat. Phys. 67, 575–608 (1992)] ten years ago. These entropy inequalities are one step in our study of the trend to equilibrium for the Boltzmann equation, in both its spatially homogeneous and spatially inhomogeneous versions.

Classification :

∗82C40 Kinetic theory of gases

76P05 Molecular or atomic structure

Zbl 1039.35147

Villani, Cedric

Optimal transportation, dissipative PDE's and functional inequalities. (English)

Caffarelli, Luis A. (ed.) et al., Optimal transportation and applications. Lectures given at the C.I.M.E. summer school, Martina Franca, Italy, September 2–8, 2001. Berlin: Springer. Lect. Notes Math. 1813, 53-89 (2003). ISBN 3-540-40192-X/pbk
http://link.springer.de/link/service/series/0304/tocs/t1813.htm

The author studies three fundamental problems in the theory of mass transportation:
Problem A (Rate of convergence as $t \to \infty$): Consider the diffusive, nonlinear partial differential equation where the unknown $(f_t)_{t\geq 0}$ is a time-dependent probability density on $\mathbb{R}^d$,

$$(*) \qquad \frac{\partial f}{\partial t} = \sigma \Delta f + \nabla \cdot (f \nabla V) + \nabla \cdot (f \nabla (f \partial W)), \qquad t \geq 0,\ x \in \mathbb{R}^d$$

(here ∇ stands for the gradient operator in $\mathbb{R}^d$, while $\nabla\cdot$ is its adjoint, the divergence operator; moreover, V and W are smooth potentials). Does this equation admit a stationary state? If the answer is yes, do solutions of $(*)$ converge towards this stationary state?
Problem B (Rate of convergence as $N \to \infty$): Consider a bunch of N particles in $\mathbb{R}^d$, with respective positions X_t^i $(1 \leq i \leq N)$ at time $t \geq 0$, solutions of the stochastic differential equation

$$dX_t^i = dB_t^i - \nabla V(X_t^i)\, dt - \frac{1}{N} \sum_{j=1}^{N} \nabla W(X_t^i - X_t^j)\, dt, \qquad 1 \leq i \leq N,$$

starting from some chaotic initial configuration $X_0 = (X_0^1, \ldots, X_0^N)$ with law $\mathcal{L}(X_0) = f_0^{\otimes N}\, dx$ on $(\mathbb{R}^d)^N$. Under some assumptions on V and W it is known that for each time $t > 0$ the density of particles in $\mathbb{R}^d$ associated with this system converges, as $N \to \infty$, towards the unique solution of $(*)$. This means for instance, that for any bounded continuous function

$$\frac{1}{N} \sum_{i=1}^{N} \varphi(X_t^i) \xrightarrow[N\to\infty]{} \int_{\mathbb{R}^d} f_t \varphi\, dx,$$

in the sense of convergence in law for random variables (note that the left-hand side is random, while the right-hand side is not). Can one estimate its speed of convergence as $N \to \infty$?
Problem C (Optimal constants): What is the optimal constant, and how do minimizers look like, in the Gagliardo-Nirenberg interpolation inequality

$$\|w\|_{L^p(\mathbb{R}^n)} \leq C \|\nabla w\|^{\theta}_{L^2(\mathbb{R}^n)} \|x\|^{1-\theta}_{L^q(\mathbb{R}^n)}$$

(with some compatibility conditions on p, q)? What about the Young inequality

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}\|g\|_{L^q(\mathbb{R}^n)}?$$

Stanislaw Wedrychowicz (Rzeszów)

Keywords : mass transportation; time-dependent probability density; stationary state; stochastic differential equation; Gagliardo-Nirenberg interpolation inequality; Young inequality

Classification :

∗35R60 PDE with randomness
60H15 Stochastic partial differential equations
82C70 Transport processes

Zbl 1013.00028

Ambrosio, Luigi; Brenier, Yann; Buttazzo, Giusseppe; Caffarelli, Luis A.; Salsa, S. (ed.); Villani, Cédric

Optimal transportation and applications. Lectures given at the C. I. M. E. summer school, Martina Franca, Italy, September 2–8, 2001. (English)

Lecture Notes in Mathematics. 1813. Berlin: Springer. viii, 169 p. EUR 27.95/net; sFr. 48.00; £19.50; $ 36.80 (2003). ISBN 3-540-40192-X/pbk
http://link.springer.de/link/service/series/0304/tocs/t1813.htm

The articles of this volume will be reviewed individually.

Keywords : Optimal transportation; Summer school; Martina Franca (Italy); CIME

Classification :

∗00B25 Proceedings of conferences of miscellaneous specific interest
49-06 Proceedings of conferences (calculus of variations)

Zbl 1170.82369

Villani, Cédric

A review of mathematical topics in collisional kinetic theory. (English)

Friedlander, S. (ed.) et al., Handbook of mathematical fluid dynamics. Vol. 1. Amsterdam: Elsevier. 71-305 (2002). ISBN 0-444-50330-7

Summary: This paper reviews some of the major macroscopic and kinetic mathematical models of vehicular traffic available in the specialized literature. Application to road networks is also discussed, and an extensive list of references is provided to both the original works presented here and other sources for further details. Microscopic models are only briefly recalled in the Introduction for the sake of thoroughness, as they are actually beyond the scope of the review. However, suitable references are included for the interested readers.

Classification :

∗82C40 Kinetic theory of gases

35F20 General theory of first order nonlinear PDE
76P05 Molecular or atomic structure
82-02 Research monographs (statistical mechanics)

Zbl 1128.82303

Villani, C.

On the trend to equilibrium for kinetic equations. (English)
Markov Process. Relat. Fields 8, No. 2, 237-250 (2002). ISSN 1024-2953
http://www.math.msu.su/ malyshev/mprf.htm

Summary: We summarize recent works about the convergence to thermodynamical equilibrium for the Boltzmann equation modelling dilute gases.

Keywords : kinetic equations; Boltzmann equation; long-time behavior; logarithmic Sobolev inequalities
Classification :
∗82C40 Kinetic theory of gases
76P05 Molecular or atomic structure

Zbl 1119.82037

Villani, Cédric

Hydrodynamic limits of Boltzmann's equation (following C. Bardos, F. Golse, C. D. Levermore, P.-L. Lions, N. Masmoudi, L. Saint-Raymond). (Limites hydrodynamiques de l'équation de Boltzmann (d'après C. Bardos, F. Golse, C. D. Levermore, P.-L. Lions, N. Masmoudi, L. Saint-Raymond).) (French)
Bourbaki seminar. Volume 2000/2001. Exposés 880-893. Paris: Société Mathématique de France. Astérisque 282, 365-405, Exp. No. 893 (2002). ISBN 2-85629-130-9/pbk
http://www.umpa.ens-lyon.fr/ cvillani/Cedrif/B02.Bourbaki.pdf
numdam:SB$_2$000 – 2001$_{4\,3_{3\,65_0}}$

Summary: This is the text of the Bourbaki seminar which I gave in June 2001 to present the works of Claude Bardos, François Golse, Dave Levermore, Pierre-Louis Lions, Nader Masmoudi and Laure Saint-Raymond about the incompressible hydrodynamical limit of the Boltzmann equation. The joint efforts of these people led to one of the most impressive mathematical achievements of these last years in fluid mechanics: a rigorous and general proof that solutions of the Boltzmann equation do reduce, in a certain physical regime, to solutions of the incompressible Navier-Stokes equations. Even though the results can be hoped to be improved in many respects (these are only statements about rescaled sequences of solutions, the proof is not constructive and does not cover physically realistic interactions), these contributions are an important step towards the accomplishment of one of the dreams of Maxwell, Boltzmann and Hilbert: the derivation of the basic equations of fluid mechanics from atomistic models. Besides, they construct a bridge between two of the most famous theories of weak solutions in fluid

mechanics: the DiPerna-Lions and Leray theories.
Written in French, this text tries to explain the main ideas and results to a non-expert reader. After an elementary introduction to the problem and its motivations, I describe the Bardos-Golse-Levermore program as it was formalized ten years ago; then I sketch several important estimates including entropy estimates, the analysis of acoustic waves, and weak compactness methods via the "regularity of the gain operator" and "averaging lemmas". (From the author's web page)

Keywords : Boltzmann; Euler; Stokes and Navier-Stokes equations; hydrodynamic limit; entropy
Classification :
∗82C40 Kinetic theory of gases
35F20 General theory of first order nonlinear PDE
35Q35 Other equations arising in fluid mechanics
45K05 Integro-partial differential equations
76A02 Foundations of fluid mechanics
76P05 Molecular or atomic structure

Zbl 1092.82032

Desvillettes, L.; Villani, Cédric
On a variant of Korn's inequality arising in statistical mechanics. (English)
ESAIM, Control Optim. Calc. Var. 8, 603-619 (2002). ISSN 1292-8119; ISSN 1262-3377
http://dx.doi.org/10.1051/cocv:2002036
numdam:$\mathrm{COCV}_2002_{8_6{}^03_0}$
http : //www.edpsciences.org/journal/index.cfm?edpsname = cocv

Summary: We state and prove a Korn-like inequality for a vector field in a bounded open set of $\mathbb{R}^N$, satisfying a tangency boundary condition. This inequality, which is crucial in our study of the trend towards equilibrium for dilute gases, holds true if and only if the domain is not axisymmetric. We give quantitative, explicit estimates on how the departure from axisymmetry affects the constants; a Monge-Kantorovich minimization problem naturally arises in this process. Variants in the axisymmetric case are briefly discussed.

Keywords : Korn's inequality; Boltzmann equation; Monge-Kantorovich mass transportation problem
Classification :
∗82C40 Kinetic theory of gases
76P05 Molecular or atomic structure
49Q20 Variational problems in geometric measure-theoretic setting

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Zbl 1053.74012

Caglioti, E.; Villani, C.
Homogeneous cooling states are not always good approximations to granular flows. (English)

Arch. Ration. Mech. Anal. 163, No. 4, 329-343 (2002). ISSN 0003-9527; ISSN 1432-0673
http://dx.doi.org/10.1007/s002050200204
http://www.springerlink.com/content/101155/

Summary: A widespread belief in the study of granular flow is the existence of "homogeneous cooling states", i.e., self-similar solutions which would attract all solutions, faster than the equilibrium solution does. In most cases, the existence of these self-similar solutions is an open problem. Here we consider a one-dimensional model, which has been used for some years, and for which simple self-similar solutions do exist. However, we prove that the approximation is quite poor. Our proof makes use of the powerful and simple tools of mass transportation, and exploits the structure of the evolution equation, seen as a nonlinear transport equation.

Keywords : self-similar solutions; nonlinear transport equation
Classification :
∗74E20 Granularity
76T25 Granular flows
82C21 Dynamic continuum models

Zbl 1032.76065

Gamba, Irene M.; Panferov, Vladislav; Villani, Cedric

On the inelastic Boltzmann equation with diffusive forcing. (English)

Birman, Michael Sh. (ed.) et al., Nonlinear problems in mathematical physics and related topics II. In honour of Professor O. A. Ladyzhenskaya. New York, NY: Kluwer Academic Publishers. Int. Math. Ser., N.Y. 2, 179-192 (2002). ISBN 0-306-47333-X/hbk; ISBN 0-306-47422-0

Summary: We discuss recent results on the study of inelastic homogeneous Boltzmann equation for hard spheres, with a diffusive term representing a random background acceleration. We show that the initial value problem has a unique solution which becomes infinitely smooth and rapidly decaying after a short time under the assumption that the data is in $L^2(\mathbb{R}^3) \cup L^1_2(\mathbb{R}^3)$ (bounded mass and energy). In addition, the time-dependent solution converges, along a subsequence of times, to a stationary solution. We also show that the high-velocity tails of both stationary and time-dependent particle distribution functions are overpopulated with respect to Maxwellian distribution, and we estimate the solutions from below.

Keywords : convergence; regularity; Gronwall estimates; parabolicity; diffusion operator; collision term; Povzner inequalities; hard spheres; random background acceleration; initial-value problem; unique solution; time-dependent solution; Maxwellian distribution
Classification :
∗76P05 Molecular or atomic structure

Zbl 1029.82036

Alexandre, R.; Villani, C.
On the Boltzmann equation for long-range interactions. (English)
Commun. Pure Appl. Math. 55, No.1, 30-70 (2002). ISSN 0010-3640
http://dx.doi.org/10.1002/cpa.10012

A.Balint (Timişoara)
Classification :
∗82C40 Kinetic theory of gases
35F20 General theory of first order nonlinear PDE
49J20 Optimal control problems with PDE (existence)

Zbl 1134.35312

Otto, Felix; Villani, Cédric
Comment on "Hypercontractivity of Hamilton-Jacobi equations" by S. G. Bobkov, I. Gentil and M. Ledoux. (English)
J. Math. Pures Appl. (9) 80, No. 7, 697-700 (2001). ISSN 0021-7824
http://dx.doi.org/10.1016/S0021-7824(01)01207-7
http://www.sciencedirect.com/science/journal/00217824

The authors prove an HWI inequality for the diffusion semigroup on $\mathbb{R}^d$ generated by $\Delta - \nabla V$ where V is a convex smooth function. From this they derive an inverse transportation cost inequality first given by *S. Bobkov, I. Gentil, M. Ledoux* [ibid. 80, No. 7, 669–696 (2001; Zbl 1038.35020)] for smooth V with Hessian matrix bounded from below.

Olaf Ninnemann (Berlin)
Classification :
∗35F20 General theory of first order nonlinear PDE
47D07 Markov semigroups of linear operators
47N20 Appl. of operator theory to differential and integral equations
49L20 Dynamic programming method (infinite-dimensional problems)
60J35 Transition functions

Zbl 1038.82073

Desvillettes, L.; Villani, C.
Entropic methods for the study of the long time behavior of kinetic equations. (English)
Transp. Theory Stat. Phys. 30, No. 2-3, 155-168 (2001). ISSN 0041-1450; ISSN 1532-2424
http://dx.doi.org/10.1081/TT-100105366
http://taylorandfrancis.metapress.com/openurl.asp?genre=journalissn=0041-1450

Summary: We show how the use of the concepts of entropy and entropy dissipation helps to find estimates of return to the equilibrium in the context of kinetic equations with a collision kernel. This approach enables one to give explicit constants in those estimates.

Classification :

∗82C40 Kinetic theory of gases
76P05 Molecular or atomic structure
82C31 Stochastic methods in time-dependent statistical mechanics

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Zbl 1029.82032

Desvillettes, L.; Villani, C.

On the trend to global equilibrium in spatially inhomogeneous entropy-dissipating systems: The linear Fokker-Planck equation. (English)

Commun. Pure Appl. Math. 54, No.1, 1-42 (2001). ISSN 0010-3640
http://dx.doi.org/10.1002/1097-0312(200101)54:1¡1::AID-CPA1¿3.0.CO;2-Q

This is the first in a series of two papers exposing a general method to overcome the problem of infinitely many equilibria. It is based on log-Sobolev inequalities and entropy. This article explains clearly the method and applies it for simplicity of presentation to a linear Fokker-Planck equation of the type

$$\partial_t f + v\nabla_x f - \nabla V(x)\cdot\nabla_v f = \nabla_v(\nabla_v f + fv)$$

where $f(t,x,v) \geq 0$ is for fixed t a probability density, and V is a smooth potential strictly convex at ∞.

The main result proves a decay to equilibrium, which is faster that $\mathcal{O}(t^{-1/\varepsilon})$ for any $\varepsilon > 0$. This is not optimal for their example as the authors point out themselves. In a second article [Preprint (2003)] the method is applied to nonlinear problems of Boltzmann-type.

The article also contains a well written general overview of the problem and an extensive reference list.

Dirk Blömker (Aachen)

Keywords : Fokker-Planck equation; logarithmic Sobolev inequality; entropy; Boltzmann equation; degenerate collision operator; kinetic equations

Classification :

∗82C40 Kinetic theory of gases
35Q99 PDE of mathematical physics and other areas
46E99 Linear function spaces and their duals
47N55 Appl. of operator theory in statistical physics
60H99 Stochastic analysis
60J99 Markov processes
82C70 Transport processes

Zbl 1139.82326

Markowich, P. A.; Villani, C.

On the trend to equilibrium for the Fokker-Planck equation: an interplay between physics and functional analysis. (English)

Mat. Contemp. 19, 1-29 (2000). ISSN 0103-9059
http://www.mat.unb.br/ matcont/

The paper presents connections between the problem of trend to equilibrium for the Fokker-Planck equation and the several inequalities from functional analysis. These inequalities are formulated in terms of relative entropy of probability measures and relative Fisher information. Such inequalities like logarithmic Sobolev or Poincaré inequalities as well as the inequalities arising in the context of concentration of measures or in the study of Gaussian isoperimetry are applied to the problems under consideration.
Various examples illustrates the general statements. Among them are generalized porous medium equations, Monge-Ampère and Hamilton-Jacobi equations, the sticky particle system.

Nikita E. Ratanov (Bogotá)

Keywords : Fokker-Planck equation; trend to equilibrium; generalized porous medium equations; sticky particle system; logarithmic Sobolev inequalities; Poincaré inequality
Classification :
∗82C31 Stochastic methods in time-dependent statistical mechanics
35B40 Asymptotic behavior of solutions of PDE

Zbl 1034.82032

Toscani, G.; Villani, C.

On the trend to equilibrium for some dissipative systems with slowly increasing a priori bounds. (English)

J. Stat. Phys. 98, No. 5-6, 1279-1309 (2000). ISSN 0022-4715; ISSN 1572-9613
http://dx.doi.org/10.1023/A:1018623930325
http://www.springerlink.com/openurl.asp?genre=journalissn=0022-4715

The authors prove the convergence to equilibrium with explicit rates for various kinetic equations including the Fokker-Planck equation and the Landau and Boltzmann equations with soft potentials. The tools that are used come from probability and include logarithmic Sobolev type inequalities. They can cover situations in which compactness arguments, more classical in partial differential equation analysis, do not apply.

Eric Sonnendrücker (Strasbourg)

Classification :
∗82C05 Classical dynamic and nonequilibrium statistical mechanics (general)
82C31 Stochastic methods in time-dependent statistical mechanics
82C40 Kinetic theory of gases

Zbl 1010.82023

Villani, C.

Decrease of the Fisher information for solutions of the spatially homogeneous Landau equation with Maxwellian molecules. (English)

Math. Models Methods Appl. Sci. 10, No.2, 153-161 (2000). ISSN 0218-2025
http://dx.doi.org/10.1142/S0218202500000100
http://www.worldscinet.com/m3as/m3as.shtml

Summary: We give a direct proof of the fact that, in any dimension of the velocity space, Fisher's quantity of information is nonincreasing with time along solutions of the spatially homogeneous Landau equation for Maxwellian molecules. This property, which was first seen in numerical simulation in plasma physics, is linked with the theory of the spatially homogeneous Boltzmann equation.

Keywords : Fisher's quantity of information; spatially homogeneous Landau equation
Classification :
∗82C05 Classical dynamic and nonequilibrium statistical mechanics (general)

Zbl 0994.94018

Villani, Cédric

A short proof of the "concavity of entropy power". (English)

IEEE Trans. Inf. Theory 46, No.4, 1695-1696 (2000). ISSN 0018-9448
http://dx.doi.org/10.1109/18.850718
http://ieeexplore.ieee.org/xpl/RecentIssue.jsp?puNumber=18

From the text: Let f be a probability measure on $\mathbb{R}^n$. Define the action of the heat semigroup $(P_t)_{t\geq 0}$ on f by the solution of the partial differential equation

$$\frac{\partial}{\partial t} P_t f = \Delta(P_t f).$$

Equivalently, $P_t f$ is the convolution of f with the n-dimensional Gaussian density having mean vector 0 and covariance matrix $2tI_n$, where I_n is the identity matrix. The "concavity of entropy power" theorem states that

$$\frac{d^2}{dt^2} N(P_t f) \leq 0,$$

where

$$N(f) = \frac{e^{\frac{2H(f)}{n}}}{2\pi e}, \quad H(f) = -\int_{\mathbb{R}^n} f \log f.$$

The functional $N(f)$ is the so-called "entropy power" of f, as introduced by Shannon, while $H(f)$ is Shannon's entropy functional. The author gives a simple proof of the concavity of the entropy power theorem.

Keywords : Fisher information; heat semigroup; entropy power

Classification :

∗94A17 Measures of information
60J35 Transition functions
62B10 Statistical information theory

Zbl 0985.58019

Otto, F.; Villani, C.
Generalization of an inequality by Talagrand and links with the logarithmic Sobolev inequality. (English)
J. Funct. Anal. 173, No.2, 361-400 (2000). ISSN 0022-1236
http://dx.doi.org/10.1006/jfan.2000.3557
http://www.sciencedirect.com/science/journal/00221236

Summary: The authors show that transport inequalities, similar to the one derived by *M. Talagrand* [Geom. Funct. Anal. 6, 587-600 (1996; Zbl 0859.46030)] for the Gaussian measure, are implied by logarithmic Sobolev inequalities. Conversely, Talagrand's inequality implies a logarithmic Sobolev inequality if the density of the measure is approximately loc-concave, in a precise sense. All constants are independent of the dimension and optimal in certain cases.
The proofs are based on partial differential equations and an interpolation inequality involving the Wasserstein distance, the entropy functional, and the Fisher information.

Keywords : logarithmic Sobolev inequalities; Talagrand's inequality
Classification :

∗58J65 Diffusion processes and stochastic analysis on manifolds
28A35 Measures and integrals in product spaces
60E15 Inequalities in probability theory
60G15 Gaussian processes

Zbl 0968.76076

Alexandre, R.; Desvillettes, L.; Villani, C.; Wennberg, B.
Entropy dissipation and long-range interactions. (English)
Arch. Ration. Mech. Anal. 152, No.4, 327-355 (2000). ISSN 0003-9527; ISSN 1432-0673
http://dx.doi.org/10.1007/s002050000083
http://www.springerlink.com/content/101155/

Summary: We study Boltzmann collision operator for long-range interactions, i.e., without Grad's angular cut-off assumption. We establish a functional inequality showing that the entropy dissipation controls smoothness of the distribution function, in a precise sense. Our estimate is optimal and gives a unified treatment of both the linear and nonlinear cases. We also give simple and self-contained proofs of several useful

results that were scattered in previous works. As an application, we obtain estimates for Cauchy problem, and for the Landau approximation in plasma physics.

Keywords : smoothness of distribution function; Boltzmann collision operator; long-range interactions; entropy dissipation; estimates; Cauchy problem; Landau approximation; plasma physics
Classification :
*76P05 Molecular or atomic structure
76X05 Plasmic flows
82B40 Kinetic theory of gases

Zbl 0951.35130

Desvillettes, Laurent; Villani, Cédric

On the spatially homogeneous Landau equation for hard potentials. II: H-theorem and applications. (English)

Commun. Partial Differ. Equations 25, No.1-2, 261-298 (2000). ISSN 0360-5302; ISSN 1532-4133
http://dx.doi.org/10.1080/03605300008821513
http://taylorandfrancis.metapress.com/openurl.asp?genre=journalissn=0360-5302

The paper is devoted to the spatially Landau equation

$$\partial f/\partial t(t,v) = Q(f,f)(t,v) \quad (v \in \mathbb{R}^N,\ t \geq 0),$$

where f is a nonnegative function and Q is a nonlinear quadratic operator acting on the variable v only. A lower bound for the entropy dissipation of the spatially homogeneous Landau equation with hard potentials in terms of the entropy is found. Explicit estimates on the speed of convergence towards equilibrium for the solution of this equation are shown. In the case of so-called over-Maxwellian potentials, the convergence is exponential. A lower bound for the spectral gap of the associated linear operator in this setting is computed as well.

Dimitar A.Kolev (Sofia)

Keywords : Landau equation; Boltzmann equation; Maxwellian function; over-Maxwellian molecules; entropy dissipation
Classification :
*35Q72 Other PDE from mechanics
82C31 Stochastic methods in time-dependent statistical mechanics

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Zbl 0946.35109

Desvillettes, Laurent; Villani, Cédric

On the spatially homogeneous Landau equation for hard potentials. I: Existence, uniqueness and smoothness. (English)

Commun. Partial Differ. Equations 25, No.1-2, 179-259 (2000). ISSN 0360-5302; ISSN 1532-4133

http://dx.doi.org/10.1080/03605300008821512
http://taylorandfrancis.metapress.com/openurl.asp?genre=journalissn=0360-5302

The authors of this interesting paper study the Cauchy problem for the spatially homogeneous Landau equation (also called Fokker-Planck-Landau) in the case of hard potentials, $\partial f/\partial t = Q(f,f)$, where $f(t,v) \geq 0$ is the density of particles which at time $t \in \mathbb{R}^+$ have velocity $v \in \mathbb{R}^N$ $(N \geq 2)$. The kernel $Q(f,f)$ is a quadratic nonlocal operator acting only on the v variable and modelling the effect of the (grazing) collisions between particles. The kernel $Q(f,f)$ depends on the so-called hard potential $\psi(|z|)$ with polynomial rate having the form $\Lambda|z|^{\gamma+2}$ $(\Lambda > 0,\ \gamma \in (0,1])$. This interesting case corresponds to interactions with inverse s-power forces for $s > 2N-1$. A detailed discussion of the Cauchy problem for the considered equation is given. Some interesting qualitative properties of the solutions are considered as well. The main results can be summarized in the following way: under the hypotheses on the initial data $f_{in} \in L^1_{2+\delta} \cap L\log L(\mathbb{R}^3)$ $(\delta > 0)$ there is a unique (weak) solution f to the considered equation. A refinement of this result for $f_{in} \in L^1_{2+\delta}(\mathbb{R}^3)$ is shown as well. A theorem for uniqueness of the weak solution is proved provided that $f_{in} \in L^2_s(\mathbb{R}^3_v)$ with $s > 5\gamma + 15$.

Dimitar Kolev (Sofia)

Keywords : Landau equation; Fokker-Planck equation; Cauchy problem; unique (weak) solution
Classification :
∗35Q72 Other PDE from mechanics
82C31 Stochastic methods in time-dependent statistical mechanics

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Zbl 0958.82044

Toscani, G.; Villani, C.

Probability metrics and uniqueness of the solution to the Boltzmann equation for a Maxwell gas. (English)

J. Stat. Phys. 94, No.3-4, 619-637 (1999). ISSN 0022-4715; ISSN 1572-9613
http://dx.doi.org/10.1023/A:1004589506756
http://www.springerlink.com/openurl.asp?genre=journalissn=0022-4715

Summary: We consider a metric for probability densities with finite variance on $\mathbb{R}^d$, and compare it with other metrics. We use it for several applications both in probability and in kinetic theory. The main application in kinetic theory is a uniqueness result for the solution of the spatially homogeneous Boltzmann equation for a gas of true Maxwell molecules.

Keywords : metric for probability densities; spatially homogeneous Boltzmann equation; Maxwell molecules
Classification :
∗82C40 Kinetic theory of gases

Zbl 0944.35066

Toscani, G.; Villani, C.

Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation. (English)

Commun. Math. Phys. 203, No.3, 667-706 (1999). ISSN 0010-3616; ISSN 1432-0916
http://dx.doi.org/10.1007/s002200050631
http://link.springer.de/link/service/journals/00220/
http://projecteuclid.org/DPubS?service=UIversion=1.0verb=Displaypage=pasthandle=euclid.cm

This interesting paper deals with the spatially homogeneous Boltzmann equation, $\partial f/\partial t(t,v) = Q(f,f)$, $t \geq 0$, $v \in \mathbb{R}^N$ $(N \geq 2)$, $f(0,.) = f_0$, $f_0 \geq 0$, $f_0 \in L^1(\mathbb{R}^N)$, where the unknown f is the probability density of particles in the velocity space, and $Q(f,f)$ is the known Boltzmann collision operator. The variable v is the velocity of the particle. A new lower bound for the entropy dissipation associated with the spatially homogeneous Boltzmann equation is derived. This bound is expressed in terms of the relative entropy with respect to the equilibrium, and thus yields a differential inequality which proves convergence towards equilibrium in relative entropy, with an explicit rate. The main result gives a considerable refinement of the analogous estimate proposed by Carlen and Carvalho (1992, 1994), under very little additional assumptions. The proof takes advantage of the structure of Boltzmann's collision operator with respect to the tensor product, and its links with Fokker-Planck and Landau equations. The known Kac model is discussed as well.

Dimitar Kolev (Sofia)

Keywords: entropy; Boltzmann equation; Fokker-Planck equations; Kac model; Ornstein-Uhlenbeck semigroup
Classification:
∗35Q35 Other equations arising in fluid mechanics
82C40 Kinetic theory of gases
76P05 Molecular or atomic structure

Zbl 0934.45010

Villani, Cédric

Regularity estimates via the entropy dissipation for the spatially homogeneous Boltzmann equation without cut-off. (English)

Rev. Mat. Iberoam. 15, No.2, 335-352 (1999). ISSN 0213-2230
http://projecteuclid.org/rmi
http://www.uam.es/departamentos/ciencias/matematicas/ibero/irevista.htm

A particular type of the Boltzmann equation from the kinetic theory of gases is considered. The Boltzmann collision operator contains a kernel which is usually supposed to be locally integrable. Looking for smoothness estimates and according to physical

point of view, the author considers it to be bounded from below by a singular (otherwise smooth) function in the origin $O(1+\nu$ is the singularity order). If the entropy dissipation functional is finite; there the solution f satisfies $\sqrt{f} \in H^{\nu/2}_{\text{loc}}$.

C.Marinov (Bucureşti)

Keywords : regularity estimates; entropy dissipation; Boltzmann equation; kinetic theory of gases; Boltzmann collision operator
Classification :
∗45K05 Integro-partial differential equations
82C40 Kinetic theory of gases

Zbl 0919.35140

Villani, Cédric

Conservative forms of Boltzmann's collision operator: Landau revisited. (English)

M2AN, Math. Model. Numer. Anal. 33, No.1, 209-227 (1999). ISSN 0764-583X; ISSN 1290-3841
http://dx.doi.org/10.1051/m2an:1999112
http://publish.edpsciences.org/abstract/m2an/v33/p209
http://www.edpsciences.org/m2an/

Summary: We show that Boltzmann's collision operator

$$Q(f,f) = \frac{1}{2}\int_{\mathbb{R}^N} dv_* \int_{S^{N-1}} d\omega B(v-v_*,\omega)(f'f'_* - ff_*)$$

can be written explicitly in divergence and double divergence forms. These conservative formulations may be of interest for both theoretical and numerical purposes. We give an application to the asymptotics of grazing collisions.

Keywords : Boltzmann's collision operator; asymptotics of grazing collisions
Classification :
∗35Q72 Other PDE from mechanics
82C40 Kinetic theory of gases
76P05 Molecular or atomic structure

Zbl 0957.82029

Villani, C.

On the spatially homogeneous Landau equation for Maxwellian molecules. (English)

Math. Models Methods Appl. Sci. 8, No.6, 957-983 (1998). ISSN 0218-2025
http://dx.doi.org/10.1142/S0218202598000433
http://www.worldscinet.com/m3as/m3as.shtml

From the introduction: First the author obtains a simplified expression for the Lan-

dau equation with Maxwellian molecules; after a brief discussion of the special case of isotropic distributions, where explicit solutions are easily available, he studies the form of the collision operator in the general case, then turns to the Cauchy problem associated to the equation, and insists on its regularizing properties. This leads him to rewrite the Landau collision operator as the sum of several operators. Finally, some qualitative features of the solutions, as the decay to equilibrium, the temperature tails and the positivity are given and a family of particular self-similar solutions is exhibited.

Keywords : Fokker-Planck equation; Boltzmann equation; Landau equation; Maxwellian molecules; collision operator; Cauchy problem
Classification :
∗82C40 Kinetic theory of gases
47N20 Appl. of operator theory to differential and integral equations
76P05 Molecular or atomic structure

Zbl 0918.60093

Villani, C.

Fisher information estimates for Boltzmann's collision operator. (English)

J. Math. Pures Appl., IX. Sér. 77, No.8, 821-837 (1998). ISSN 0021-7824
http://dx.doi.org/10.1016/S0021-7824(98)80010-X
http://www.sciencedirect.com/science/journal/00217824

Summary: We derive several estimates for Boltzmann's collision operator in terms of Fisher's information. In particular, we prove that Fisher's information is decreasing along solutions of the Boltzmann equation with Maxwellian cross-section, in any dimension of velocity space, thus generalizing results by G. Toscani, E. Carlen and M. Carvalho.

Keywords : Boltzmann's collision operator; Fisher's information; Boltzmann equation; Maxwellian cross-section; velocity space
Classification :
∗60K40 Physical appl. of random processes
60H99 Stochastic analysis
94A15 General topics of information theory

Zbl 0912.45011

Villani, Cédric

On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. (English)

Arch. Ration. Mech. Anal. 143, No.3, 273-307 (1998). ISSN 0003-9527; ISSN 1432-0673
http://dx.doi.org/10.1007/s002050050106
http://www.springerlink.com/content/101155/

Author's summary: The paper deals with the spatially homogeneous Boltzmann equation when grazing collisions are involved. We study in a unified setting the Boltzmann equation without cut-off, the Fokker-Planck-Landau equation, and the asymptotics of grazing collisions for a very broad class of potentials; in particular, we are able to derive rigorously the Landau equation for the Coulomb potential. In order to do so, we introduce a new definition of weak solutions, based on entropy production.

V. Yurko (Saratov)

Keywords: Boltzman equation; Fokker-Planck-Landau equation; weak solutions; asymptotics; Coulomb potential

Classification:

∗45K05 Integro-partial differential equations
45M05 Asymptotic theory of integral equations
82C40 Kinetic theory of gases
76P05 Molecular or atomic structure

Zbl 0856.35020

Villani, Cédric

On the Cauchy problem for Landau equations: Sequential stability, global existence. (English)

Adv. Differ. Equ. 1, No.5, 793-816 (1996). ISSN 1079-9389
http://www.aftabi.com/ADE/ade.html

This paper deals with the non-homogeneous Landau (or Fokker-Planck) equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f,f), \quad t \geq 0, \quad x \in \mathbb{R}^N, \quad v \in \mathbb{R}^N,$$

$(N \geq 1, f \geq 0)$ with

$$Q(f,f) = \sum_{i,j=1}^{N} \frac{\partial}{\partial v_i} \left\{ \int_{\mathbb{R}^N} dv_* a_{ij}(v - v_*) \left[f(v_*) \frac{\partial f(v)}{\partial v_j} - f(v) \frac{\partial f(v_*)}{\partial v_{*,j}} \right] \right\},$$

where (a_{ij}) is symmetric, nonnegative, even in z and such that $\sum_{i,j} a_{ij}(z) z_i z_j = 0$ for almost every z. The main result of the paper is an existence result of renormalized solutions with a defect measure for the Cauchy problem under technical assumptions that include the Coulomb case. The result relies on a compactness property (based on a priori estimates) previously obtained by P.-L. Lions and on a detailed study of the collision operator Q, which provide a weak stability result for sequences of solutions.
The author also introduces the notion of quasi-renormalized solutions i.e. renormalized solutions with a defect measure such that it vanishes as the parameter of the renormalization tends to zero, and gives two sufficient conditions for a solution to be quasi-renormalized.
The long time behaviour is also studied using a Boltzmann's H-theorem, which is established in the paper: Renormalized solutions with a defect measure converge for large times – up to the extraction of a subsequence – to a Maxwellian function (but no asymptotic uniqueness result holds since the conservation of the energy is not established as in

Boltzman's equation). This paper uses various techniques of the renormalized solutions that have been introduced by R. J. DiPerna and P.-L. Lions. The notion of renormalized solution is generalized and adapted to the Landau equation and a very careful analysis of the collision operator is provided.

J.Dolbeault (Paris)

Keywords : Fokker-Planck equation; quasi-renormalized solutions; defect measure; Landau equation

Classification :

∗35D05 Existence of generalized solutions of PDE
82C40 Kinetic theory of gases
82D10 Plasmas

Zbl 0870.35017

Lions, Pierre-Louis; Villani, Cédric

Optimal regularity for square roots. (Régularité optimale de racines carrées.)

(French. Abridged English version)
C. R. Acad. Sci., Paris, Sér. I 321, No.12, 1537-1541 (1995). ISSN 0764-4442

Summary: Let a be a symmetric $N \times N$ matrix function on $\mathbb{R}^d$ with bounded second derivatives, then we have

$$|Da^{1/2}|_{L^\infty(\mathbb{R}^d)} \leq C|D^2 a|^{1/2}_{L^\infty(\mathbb{R}^d)},$$

where C depends only on N and d. We show that this result extends in a natural way to the case when a has second-order derivatives in L^d, $1 < p \leq \infty$: in particular, we show, when $d = 1$, $|(a^{1/2})'(x)| \leq C[M(|a''|)]^{1/2}(x)$, a.e. $x \in \mathbb{R}$, where C depends only on N, and $M(u)$ denotes the maximal function of u. This pointwise inequality will allow us to show that if $D^2 a \in L^p$ then $Da^{1/2} \in L^{2p}$ and $|Da^{1/2}|_{L^{2p}} \leq C|D^2 a|_{L^p}$, where C depends only on p, d and N. Such results are useful for the study of the Fokker-Planck-Landau equation.
Finally, we present an extension of the above estimates: Let a be a symmetric $N \times N$ matrix function such that $a \geq 0$ a.e., and $a \in \dot{W}^{s,p}(\mathbb{R}^d)$, $1 \leq p \leq \infty$, $0 < s < 1$, then $a^{1/(1+s)} \in \dot{W}^{1,p(1+s)}$.

Keywords : Fokker-Planck-Landau equation

Classification :

∗35B65 Smoothness of solutions of PDE